Scientific Papers. Series E. Land Reclamation, Earth Observation & Surveying, Environmental Engineering. Vol II, 2013
Print ISSN 2285-6064, CD-ROM ISSN 2285-6072, ISSN-L 2285-6064

APPLIED GEOMETRY IN MICROECONOMICS. RECENT DEVELOPMENTS

Adela MIHAI[1,2], Andreea OLTEANU[3]

[1]Technical University of Civil Engineering Bucharest, [2]University of Bucharest, Romania
[3]University of Agronomic Sciences and Veterinary Medicine, Bucharest, Romania

Corresponding author e-mail: andreea.olteanu@fifim.ro, andreea_d_olteanu@yahoo.com

Abstract

Recently, B.Y. Chen (Chen, 2011, 2012 (1)) studied some geometric properties of h-homogeneous production functions with applications in microeconomics. The class of production functions includes many important production functions in microeconomics; in particular, the well-known generalized Cobb-Douglas production function, widely used in economics to represent the relationship of an output to inputs, and the ACMS production function, also known as the Armington aggregator are production functions.
In (Mihai and Sandu, 2012), the authors continued the study of geometry of h-homogeneous production functions by considering the minimality property of the production hypersurface and also the minimality of a production surface corresponding to a quasi-sum production function of 2-variables.
In (Chen, 2012 (2)), B. Y. Chen classified h-homogeneous production functions with constant elasticity of substitution.
In this paper we make a survey of recent results on production functions obtained by B.Y. Chen, especially from (Chen, 2011), and also recall the results obtained by the first author in (Mihai and Sandu, 2012).
In this note, we consider examples of known production functions and verify by concrete calculations some of the previous results. We study production surfaces by considering their constant Gauss curvature. Also, we calculate the mean curvature for some particular production function of two-factors.

Keywords: *h-homogeneous production function, perfect substitute, production hypersurface, quasi-sum production function.*

INTRODUCTION

The production function is one of the key concepts in the economic field. A *production function* is a non-constant positive function, specifying the output of a firm, an industry or even entire economy for all combinations of inputs.

There is a very important class of production functions that are often analyzed in both microeconomics and macroeconomics; namely, *h-homogeneous production functions*.
A function f of a multiple variables $x_1, x_2, ..., x_n$ is a *h-homogenous function* (or *homogenous of degree h*) if

(1.1) $f(tx_1, tx_2, ..., tx_n) = t^h f(x_1, x_2, ..., x_n)$,

for any given positive constant t and some constant h, where h is the *degree* of f.
A *h-homogenous production function* is a production function homogenous of degree h (a h-homogenous production function). This class of production functions includes many important production functions in microeconomics, such as the Cobb-Douglas production function and the generalized Cobb-Douglas production function, the ACMS production function and the generalized ACMS production function. For details about the production functions and their history, please see (Cobb and Douglas, 1928), (Douglas, 1976), (Filipe and Adams, 2005), (Mishra, 2010).

One denotes by $Q = f(x_1, x_2, ..., x_n)$ a h-homogenous production function. If $h > 1$, the function exhibits increasing return to scale; if $h < 1$, it exhibits decreasing return to scale; if $h = 1$, it exhibits constant return to scale. A homogenous function of degree 1 is often called *linearly homogenous*.

Cobb and Douglas (Cobb and Douglas, 1928) defined a 2-factor production function (CD *production function*) of the form

(1.2) $Y = bL^kC^{1-k}$,

where L represents the labor input, C is the capital input, b the total factor input and Y the total production. Douglas was looking for a function which estimates the relationship of an output to inputs of the workers and capital.

A *generalized CD production function* is of the form

(1.3) $Q(x) = bx_1^{\alpha_1}x_2^{\alpha_2}\dots x_n^{\alpha_n}$,

$x = (x_1, x_2, \dots, x_n)\epsilon \boldsymbol{R}_+^n$,

with b a positive constant and α_i, $i\epsilon\overline{1,n}$, non-zero constants. The function Q is homogenous of degree $h = \sum_{j=1}^n \alpha_j$.

Arrow, Chenery, Minhas and Solow (Arrow, Chenery, Minhas and Solow, 1961) defined another *2*-factor production function called the *ACMS – production function*,

(1.4) $Q' = F[aK^r + (1-a)L^r]^{\frac{1}{r}}$

where Q' is the output, F is the factor productivity, a is the share parameter, K, L are the primary production factors, $r = \dfrac{s-1}{s}$ and $s = \dfrac{1}{1-r}$ is the elasticity of substitution.

There is also a *generalization of ACMS production function* as follows:

(1.5) $Q'(x) = b(\sum_{i=1}^n a_i^\rho x_i^\rho)^{\frac{h}{\rho}}$,

$x = (x_1, x_2, \dots, x_n)\epsilon \boldsymbol{R}_+^n$,

where a_i, b, h, ρ are constants, $b, h > 0, \rho < 1$ and $a_i, \rho \neq 0$.

For each production function $Q = f(x_1, x_2, \dots, x_n)$ it is possible to define a non-parametric hypersurface of an Euclidean *(n+1)*–space $\boldsymbol{E}^{n+1}$ endowed with the canonical euclidian structure,

$x \cdot y = x_1y_1 + x_2y_2 + \dots + x_ny_n + x_{n+1}y_{n+1}$,

$x = (x_1, x_2, \dots, x_{n+1})$,

$y = (y_1, y_2, \dots, y_{n+1})$, $x, y \epsilon \boldsymbol{E}^{n+1}$,

given by

$L = (x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, x_n, f(x_1, x_2, \dots, x_n))$.

L is called a *production hypersurface* (Chen, 2011). If $n=2$, then L is a *production surface*.

Chen in (Chen, 2011) studied geometric properties of h-homogenous production functions via their corresponding production hypersurface. Geometric properties of CD and ACMS-production hypersurfaces and their generalizations were proved in (Chen, 2011), (Vîlcu and Vîlcu, 2011), (Glen, 2011).

A production function Q is called *quasi-sum* (Chen, 2012 (2), (3)) if there are continuous strict monotone functions $h_i : \boldsymbol{R}_+ \rightarrow \boldsymbol{R}$, $i\epsilon\overline{1,n}$ and there exist an interval $I \subset \boldsymbol{R}$ of positive lenght and a continuous strict monotone function $F : I \rightarrow \boldsymbol{R}_+$ such that for each $x\epsilon\boldsymbol{R}_+^n$ we have $h_1(x_1) + \dots h_n(x_n) \epsilon I$ and

(1.6) $Q = f(x_1, x_2, \dots, x_n) =$
$= F(h_1(x_1) + \dots + h_n(x_n))$.

The quasi-sum production functions are related to the problem of consistent aggregation (Aczel, 1996). The generalized CD production functions (1.3) and the ACMS production functions (1.4) are examples of quasi-sum production functions. A quasi-sum product function is *quasi-linear* if at most one of $F, h_1, \dots, h_n$ in (1.6) is a non-linear function.

Let M be a hypersurface in $\boldsymbol{E}^{n+1}$. For general references on the geometry of hypersurfaces please see (Chen, 1973).

Recall the following notations on M: ξ is the *unit normal* at M; g is the *metric tensor*, having the coefficients $g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$ with (g^{ij}) the *inverse matrix* of (g_{ij}); dV is the volume element; h_{ij} are the coefficients of the second fundamental form h; H is the mean curvature vector; G is the Gauss-Kronecker curvature.

In (Chen, 2011, 2012 (1), (2)) B. Y. Chen gives the known formulas for the previous quantities. More precisely, the following statements hold:

Proposition 1.1 (Chen, 2011, 2012 (1), (2)) For the production hypersurface M of $\boldsymbol{E}^{n+1}$, defined by

(1.7) $L(x_1, \dots, x_n) = (x_1, \dots, x_n, f(x_1, \dots, x_n))$,

with $Q = f(x_1, x_2, \dots, x_n)$ a h-homogenous production function and $w = \sqrt{1 + \sum_{i=1}^n f_i^2}$ where $f_i = \frac{\partial f}{\partial x_i}, i = 1, \dots n$, we have:

(1.8) $\xi = -\frac{1}{w}(f_1, \dots f_n, -1)$, (1.9) $g_{ij} = \delta_{ij} + f_i f_j$, where $\delta_{ij} = \begin{cases} 1, if\ i = j \\ 0, if\ i \neq j \end{cases}$ is the Kronecker symbol.

(1.10) $dV = \sqrt{g_{ij}}\, dx_1 \wedge \dots \wedge dx_n = w dx_1 \wedge \dots \wedge dx_n$,

(1.11) $g^{ij} = \delta_{ij} - \frac{1}{w^2} f_i f_j$,

(1.12) $h_{ij} = \frac{1}{w} f_{ij}, where\ f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$,

(1.13) $H = \frac{1}{n} \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\frac{1}{w} f_j \right)$,

(1.14) $G = \frac{\det(h_{ij})}{\det(g_{ij})} = \frac{1}{w^{n+2}} \det(f_{ij})$.

We point out the geometric interpretation of the geometric quantities from above.

The *Gauss-Kronecker curvature* measures how far is a hypersurface for being flat. When $n = 2$, the Gauss-Kronecker curvature is simply called the *Gauss curvature*, which is an intrinsec invariant (depends on the surface M only). A surface of null Gauss curvature is a *flat surface*.

The *mean curvature vector H* measures the tension received by the hypersurface from the ambient (Euclidian space). A hypersurface of null mean curvature is *minimal*. Of all hypersurfaces with a given boundary, *the minimal one* has maximum volume.

2. RECENT RESULTS ON *h*-HOMOGENEOUS AND QUASI-SUM PRODUCTION FUNCTIONS

A production function is a *perfect substitute* (Chen, 2011) if it is *1*-homogeneous (linearly homogeneous) which takes the form

(2.1) $Q(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} a_i x_i$,

for some non-zero constant $a_1, \dots, a_n$.
A perfect substitute with inputs capital and labour has the property that the marginal and average physical products of both capital and labour can be expressed as functions of the capital-labour ratio alone (see (Chen, 2011)).

Denote by $\boldsymbol{R}_+ = \{t \in \boldsymbol{R},\ t > 0\}$ and by $\boldsymbol{R}_+^n = \{(x_1, x_2, \dots x_n) / x_1, x_2, \dots x_n > 0\}$.

B. Y. Chen proved geometric characterization for a *h*-homogeneous production function to have constant return to scale or to be a perfect substitute.

Theorem 2.1 (Chen, 2012(3)) Let $Q = f(x_1, \dots, x_n)$ be a homogeneous production function of degree $d \neq 0$. Then the production hypersurface of Q has null Gauss-Kronecker curvature if and only if either

i) the production function has constant return to scale, or
ii) the production function is of form:

$$f(x_1, \dots, x_n) = \left(x_1 \phi\left(\frac{x_2}{x_1}, \dots, \frac{x_n}{x_1} \right) \right)^d,$$

where $\phi(u_1, \dots, u_{n-1})$ is an *(n-1)*-input function satisfying the homogeneous Monge-Ampère equation: $\det(\phi_{ij}) = 0$.

Theorem 2.2 (Chen, 2011) A *h*-homogeneous production function with more than two factors is a perfect substitute if and only if the production hypersurface is flat.

Theorem 2.3 (Chen, 2011) A two-factor *h*-homogeneous production function is a perfect substitute if and only if the production surface is a minimal surface.

The following corrolaries were proved.

Corollary 2.4 (Chen, 2011) The generalized Cobb-Douglas production function has constant return to scale if and only if the production hypersurface has null Gauss-Kronecker curvature.

Corollary 2.5 (Chen, 2011) The two-factor Cobb-Douglas production function has constant return to scale if and only if the production surface is flat.

Corollary 2.6 (Chen, 2011) The production hypersurface of the generalized Cobb-Douglas production function with more than two factors is always non-flat.

Similar results have been obtained for ACMS production functions:

Corollary 2.7 (Chen, 2011) The ACMS production function has constant return to scale if and only if the production hypersurface has null Gauss-Kronecker curvature.

Corollary 2.8 (Chen, 2011) The ACMS production function with more than *2*-factors is a perfect substitute if and only if the product hypersurface is flat.

In (Chen, 2012(1)) B. Y. Chen obtained a necessary and sufficient condition for a quasi-sum production function to be quasi-liniar and completely classified quasi-sum production functions whose production hypersurfaces have vanishing Gauss-Kronecker curvature.

Theorem 2.9 (Chen, 2012(1)) A twice – differentiable quasi-sum production function with than two-factors is quasi-liniar if and only if its production hypersurface is a flat space.

Theorem 2.10 (Chen, 2012(1)) Let f be a twice-differentiabl quasi-sum production function. Then the production hypersurface of f has vanishing Gauss-Kronecker curvature if and only if up to translations, f is one of the following:

a) $f = ax_1 + \sum_{i=2}^{n} \varphi_i(x_i)$, with a non-zero constant and $\varphi_2, \ldots \varphi_n$ are strict monotone functions;
b) $f = F(a_1x_1 + a_2x_2 + \sum_{i=3}^{n} \varphi_i(x_i))$, where a_1, a_2 are non-zero constants and F, $\varphi_3, \ldots, \varphi_n$ are strict monotone functions;
c) f is a generalized Cobb-Douglas function given by $f = \vartheta x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ for some non-zero constants ϑ, $\alpha_1, \ldots, \alpha_n$ satisfying $\sum_{i=1}^{n} \alpha_i = 1$;
d) $f = \left(\sum_{i=1}^{n} a_i x_i^{\frac{\varepsilon-1}{\varepsilon-2}}\right)^{\frac{\varepsilon-2}{\varepsilon-1}}$, where a_i, ε are constants with $a_i \neq 0$ and with $\varepsilon \neq 1,2$;
e) $f = a \cdot ln(\sum_{i=1}^{n} b_i e^{r_i x_i})$ for some non-zero constants a, b_i, r.

For quasi-sum production function with *2*-factors the following corollary was given:

Corollary 2.11 (Chen, 2012(1)) Let $f(x, y)$ be a twice-differentiable quasi-sum production function. Then the production surface is a flat surface if and only if, up to translations, f is one of the following:

a) a quasi-liniar production function;
b) f is a Cobb-Douglas function, i.e. $f = ax^r y^{1-r}$ for some non-zero constant a, r with $r \neq 1$;
c) f is an ACMS function given by $f = \left(ax^{\frac{\varepsilon-1}{\varepsilon-2}} + by^{\frac{\varepsilon-1}{\varepsilon-2}}\right)^{\frac{\varepsilon-2}{\varepsilon-1}}$, with $\varepsilon \neq 1,2$;
d) $f = a\ ln(be^{rx} + ce^{sy})$, for some non-zero constants a, b, c, r, s.

In (Mihai and Sandu, 2012) the following characterization of *h*-homogenous production functions, by considering the minimality of the corresponding production hypersurface, was obtained.

Theorem 2.12 (Mihai and Sandu, 2012) A *h*-homogenous production function which is a perfect substitute has minimal corresponding production hypersurface.

The converse of this result is partially true.

Theorem 2.13 (Mihai and Sandu, 2012) A *h*-homogeneous production function whose corresponding production hypersurface is minimal is not always a perfect substitute.

Also, in (Mihai and Sandu, 2012) the minimality of the corresponding production surface of a *2*-factor twice differentiable quasi-sum production function was studied. The authors obtained the following:

Theorem 2.14 (Mihai and Sandu, 2012) Let $f(x_1, x_2) = (x_1, x_2, F(h_1(x_1) + h_2(x_2)))$ be a twice-differentiable quasi-sum production function. Then:

a) If F, h_1 and h_2 are all linear functions, the corresponding production surface is minimal.
b) If h_1 and h_2 are linear functions, then the minimality of the corresponding production surface implies that F is also a linear function and we have again case a).
c) If F is a linear function, then the minimality of the corresponding production surface implies that h_1 and h_2 are non-linear

functions (i.e. f is a quasi-linear quasi-sum productions function).

3. NULL GAUSS CURVATURE OF KNOWN PRODUCTION FUNCTIONS

Let us first consider the two-factor Cobb-Douglas production function

(3.1) $f(x_1,x_2) = \beta {x_1}^{\alpha_1} {x_2}^{1-\alpha_1}$,

where β is a positive constant and α_1 is a non-zero constant.

Denoting by $a(x_1) = \beta {x_1}^{\alpha_1}$ and by $b(x_2) = {x_2}^{1-\alpha_1}$, we have:

$$\dot{a}(x_1) = \beta\alpha_1 {x_1}^{\alpha_1-1},$$
$$\ddot{a}(x_1) = \beta\alpha_1(\alpha_1-1){x_1}^{\alpha_1-2}.$$

$$\dot{b}(x_2) = (1-\alpha_1){x_2}^{-\alpha_1},$$
$$\ddot{b}(x_2) = (1-\alpha_1)(-\alpha_1){x_2}^{-\alpha_1-1}.$$

Recall

$$f_1 = \frac{\partial f}{\partial x_1},$$

$$f_2 = \frac{\partial f}{\partial x_2},$$

$$f_{11} = \frac{\partial^2 f}{\partial {x_1}^2},$$
$$f_{22} = \frac{\partial^2 f}{\partial {x_2}^2},$$
$$f_{12} = \frac{\partial^2 f}{\partial x_1 \partial x_2}.$$

Then it follows that

$$\begin{gathered} f_{11}f_{22} - {f_{12}}^2 = \\ = \beta\alpha_1(\alpha_1-1){x_1}^{\alpha_1-2}{x_2}^{1-\alpha_1} \cdot \\ \cdot \beta {x_1}^{\alpha_1}(1-\alpha_1)(-\alpha_1){x_2}^{-\alpha_1-1} - \\ -(\beta\alpha_1 {x_1}^{\alpha_1-1})^2[(1-\alpha_1){x_2}^{-\alpha_1}]^2 = \\ = \beta^2{\alpha_1}^2(\alpha_1-1)^2{x_1}^{2(\alpha_1-1)}{x_2}^{-2\alpha_1} - \\ -\beta^2{\alpha_1}^2(\alpha_1-1)^2{x_1}^{2(\alpha_1-1)}{x_2}^{-2\alpha_1} = 0. \end{gathered}$$

So, $k=0$, according to the formula (1.14), where k is the Gauss curvature.

Now, we consider the two-factor ACMS production function given by

(3.2) $f(x_1,x_2) = (a_1x_1 + a_2x_2)^h$.

The partial derivatives of (3.2) with respect to x_1, x_2 are the following:

$$\begin{gathered} f_1 = h(a_1x_1+a_2x_2)^{h-1}a_1, \\ f_2 = h(a_1x_1+a_2x_2)^{h-1}a_2, \\ f_{11} = h(h-1){a_1}^2(a_1x_1+a_2x_2)^{h-2}, \\ f_{22} = h(h-1){a_2}^2(a_1x_1+a_2x_2)^{h-2}, \\ f_{12} = h(h-1)a_1a_2(a_1x_1+a_2x_2)^{h-2}. \end{gathered}$$

Then we get

$$\begin{gathered} f_{11}f_{22} - {f_{12}}^2 = \\ = h^2(h-1)^2{a_1}^2{a_2}^2(a_1x_1+a_2x_2)^{2(h-2)} - \\ -h^2(h-1)^2{a_1}^2{a_2}^2(a_1x_1+a_2x_2)^{2(h-2)} \\ = 0. \end{gathered}$$

Then the Gauss curvature k is zero, according to (1.14).

The calculations from this section agree with the theoretical results of Chen (Chen, 2011).

4. PRODUCTION FUNCTIONS WITH CONSTANT GAUSS CURVATURE

In this section we investigate two-factor production functions which have constant Gauss curvature k.

It follows from (1.14) that

$$\frac{f_{11}f_{22} - f_{12}^2}{(1+f_1^2+f_2^2)^2} = k,$$

using the same notations as in the previous sections for the partial derivatives of first and second order.

Then we obtain

(4.1) $f_{11}f_{22} - f_{12}^2 = k(1+f_1^2+f_2^2)^2$.

By having in mind the examples from the Section 3, it is natural to consider the case when the production function has separable variables, i.e, f is of the form:

$$f(x_1,x_2) = a(x_1)b(x_2)\ .$$

Then

$$\begin{gathered} f_1 = \dot{a}(x_1)b(x_2), \\ f_2 = a(x_1)\dot{b}(x_2), \\ \Rightarrow \\ f_{11} = \ddot{a}(x_1)b(x_2), \\ f_{12} = \dot{a}(x_1)\dot{b}(x_2), \end{gathered}$$

$$f_{22} = a(x_1)\ddot{b}(x_2).$$

Thus, (4.1) can be rewritten as

(4.2)

$$\ddot{a}(x_1)b(x_2)a(x_1)\ddot{b}(x_2) - \dot{a}^2(x_1)\dot{b}^2(x_2) =$$
$$= k[1 + \dot{a}^2(x_1)b^2(x_2) + a^2(x_1)\dot{b}^2(x_2)]^2.$$

Case (i): $a(x_1) = constant$ or $b(x_2) = constant$.

For example, $a(x_1) = q = constant$ ($\neq 0$).
From (4.2) we get

$$0 = k[1 + q^2\dot{b}^2(x_2)]^2.$$

Therefore we obtain $k = 0$ → case studied by Chen (flat surface) or $\dot{b}^2(x_2) = -\frac{1}{q^2}$ impossible!

Case (ii): $a(x_1) \neq constant$ and $b(x_2) \neq constant$.

Suppose that a and b are linear functions:

- $a(x_1) = a_1x_1 + a_2$, $a_1 = \dot{a}(x_1) \neq 0$ (because $a(x_1) \neq constant$);
- $b(x_2) = b_1x_2 + b_2$, $b_1 = \dot{b}(x_2) \neq 0$ (because $b(x_2) \neq constant$).

In this case, $\ddot{a}(x_1) = \ddot{b}(x_2) = 0$ and from (4.2) we obtain:

$$-a_1^2b_1^2 = k[1 + a_1^2(b_1x_2 + b_2)^2 +$$
$$+(a_1x_1 + a_2)^2b_1^2]^2.$$
$$\Rightarrow$$

$k < 0$ and

$$\sqrt{-\frac{1}{k}} = \frac{1}{a_1^2b_1^2} +$$
$$\frac{(b_1x_2 + b_2)^2}{b_1^2} + \frac{(a_1x_1 + a_2)^2}{a_1^2}.$$

The last relation is equivalent with

$$\underbrace{\frac{(a_1x_1 + a_2)^2}{a_1^2} + \frac{1}{a_1^2b_1^2} - \sqrt{-\frac{1}{k}}}_{function\ \ of\ x_1} =$$
$$= \underbrace{-\frac{(b_1x_2 + b_2)^2}{b_1^2}}_{function\ \ of\ x_2}$$

which implies that $a_1 = 0$ & $b_1 = 0$, in contradiction with $a_1 \neq 0$ and $b_1 \neq 0$.

Then $a(x_1)$ and $b(x_2)$ cannot be linear functions.

Case (iii): Now, suppose that a is a linear function of x_1 and we don't know anything about *b*:

$$a(x_1) = a_1x_1 + a_2.$$

From (4.2) we get

$$-a_1^2\dot{b}^2(x_2) = k[1 + a_1^2b^2(x_2) +$$
$$+(a_1x_1 + a_2)^2\dot{b}^2(x_2)]^2.$$
$$\Rightarrow$$

$k < 0$ and

$$1 + a_1^2b^2(x_2) + (a_1x_1 + a_2)^2\dot{b}^2(x_2) =$$
$$= \sqrt{-\frac{a_1^2\dot{b}^2(x_2)}{k}}.$$

Because we are in the case $b(x_2) \neq constant$, i. e. $\dot{b}(x_2) \neq 0$, we can divide the previous relation by $\dot{b}(x_2)$:

$$\underbrace{(a_1x_1 + a_2)^2}_{function\ \ of\ x_1} =$$
$$\underbrace{= \frac{1}{\dot{b}^2(x_2)}\sqrt{-\frac{a_1^2\dot{b}^2(x_2)}{k}} - 1 - a_1^2b^2(x_2).}_{function\ \ of\ x_2}$$
$$\Rightarrow$$

$a_1 = 0$

$$\Rightarrow$$

$a(x_1) =$ constant, case already studied.

It follows that the case of constant Gauss curvature reduces to the case of null Gauss curvature.

5. THE MEAN CURVATURE OF KNOWN 2-FACTORS PRODUCTION FUNCTIONS

i) By using symbolic computation in MathCad version 14, we have obtained the following expression for the mean curvature vector of the two-factor Cobb-Douglas production function

$$f(x,y) := a\,x^{b}\,y^{1-b},$$

with a a positive constant and b a non-zero constant.

$$\frac{d}{dx} f(x,y) \rightarrow a \cdot b \cdot x^{b-1} \cdot y^{1-b}$$

$$f1(x,y) := \frac{a \cdot b \cdot x^{b-1}}{y^{b}}$$

$$\frac{d}{dy} f(x,y) \rightarrow -\frac{a \cdot x^{b} \cdot (b-1)}{y^{b}}$$

$$f2(x,y) := -\frac{a \cdot b \cdot x^{b}}{y^{b+1}}$$

Remember

$$w := \sqrt{1 + f1(x,y)^2 + f2(x,y)^2}.$$

Then, in our case:

$$w \rightarrow \sqrt{\frac{a^2 \cdot b^2 \cdot x^{2 \cdot b}}{y^{2 \cdot b+2}} + \frac{a^2 \cdot b^2 \cdot x^{2 \cdot b-2}}{y^{2 \cdot b}} + 1}$$

$$g1(x,y) := \frac{f1(x,y)}{w}$$

$$g1(x,y) \rightarrow \frac{a \cdot b \cdot x^{b-1}}{y^{b} \cdot \sqrt{\frac{a^2 \cdot b^2 \cdot x^{2 \cdot b}}{y^{2 \cdot b+2}} + \frac{a^2 \cdot b^2 \cdot x^{2 \cdot b-2}}{y^{2 \cdot b}} + 1}}$$

$$g2(x,y) := \frac{f2(x,y)}{w}$$

$$g2(x,y) \rightarrow -\frac{a \cdot b \cdot x^{b}}{y^{b+1} \cdot \sqrt{\frac{a^2 \cdot b^2 \cdot x^{2 \cdot b}}{y^{2 \cdot b+2}} + \frac{a^2 \cdot b^2 \cdot x^{2 \cdot b-2}}{y^{2 \cdot b}} + 1}}$$

Denoting by *h1(x,y)* and *h2(x,y)* the partial derivatives of *g1* with respect to *x*, respectively *g2* with respect to *y*, from the relation (1.13) we have that *H(x,y)* is the mean average of *h1(x,y)* and *h2(x,y)*.
The final value of *H(x,y)* is:

$$H(x,y) \text{ simplify } \rightarrow \frac{a \cdot b \cdot x^{b-2} \cdot \left(b \cdot x^2 + b \cdot y^2 + x^2 - y^2\right)}{2 \cdot y^{b+2} \cdot \left(\frac{a^2 \cdot b^2 \cdot x^{2 \cdot b}}{y^{2 \cdot b+2}} + \frac{a^2 \cdot b^2 \cdot x^{2 \cdot b-2}}{y^{2 \cdot b}} + 1\right)^{\frac{3}{2}}}$$

It follows that H cannot be zero, because it is obviously that the denominator cannot be zero (b cannot be in the same time *1* and *-1*).

ii) By using symbolic computation in MathCad version 14, we have obtained the following expression for the mean curvature vector of the two-factor ACMS production function

$$f(x,y) := (a \cdot x + b \cdot y)^{h},$$

where a, b and h are non-zero constants.

We write below the final result, the denominator of the mean curvature *H(x,y)* is:

$$(a \cdot x + b \cdot y)^{h} \cdot h \cdot \left(a^2 + b^2\right) \cdot (h-1).$$

It follows that $H=0$ if and only if $h=1$.

If $b=1-a$, i.e when we consider the original ACMS production function (see Arrow K. J., Chenery H. B., Minhas B. S., Solow R. M., 1961), the denominator of the mean curvature *H(x,y)* is:

$$h \cdot (h-1) \cdot \left(2 \cdot a^2 - 2 \cdot a + 1\right) \cdot (y + a \cdot x - a \cdot y)^{h-2}$$

For this example, $H=0$ if and only if $h=1$.

Then, we have proved by straightforward calculations that the production surfaces corresponding to usual *2*-factors production functions are not minimal.

This fact means that we cannot get similar results from the view-point of minimality as those obtained by Chen about Gauss curvature (flatness) (for example Corrolary 2.5 from Section 2).

AKNOWLEDGEMENTS

The first author's work was supported by the strategic grant POSDRU/89/1.5/S/58852, Project *Postdoctoral programme for training scientific researchers*, cofinanced by the European Social Found within the Sectorial Operational Program Human Resources Development 2007 – 2013.

REFERENCES

Aczel J., G. Maksa G., 1996, Solution of the rectangular m x n generalized bisymmetry equation and of the problem of consistent aggregation, J. Math. Anal. Appl. 203, 104 – 126.

Arrow K. J., Chenery H. B., Minhas B. S., Solow R. M., 1961, Capital Labour substitution and economic efficiency, Rev. Econom. Stat. 43(3), 225 – 250.

Chen B. Y., 1973, Geometry of Submanifolds, M. Dekker, New York.

Chen B. Y., 2011, On some geometric properties of h-homogeneous production functions in microeconomics, Kragujevac J. Math. 35(3), 343 – 357.

Chen B. Y., 2012(1), A note on homogeneous production models, Kragujevac J. Math. 36(1), 41-43.

Chen B. Y., 2012(2), On some geometric properties of quasi-sum production models, J. Math. Anal. Appl. 392(2), 92–199.

Chen B. Y., 2012(3), Classification of h-homogeneous production functions with constant elasticity of substitution, Tamkang J. Math. 43(2), 321–328.

Cobb C. W., Douglas P. H., 1928, A theory of production, Amer. Econ. Rev. 18, 139 – 165.

Douglas P. H., 1976, The Cobb-Douglas production function once again: its history, its testing and some new empirical values, Journal of Political Economy 84(5), 903 – 916.

Filipe J., Adams G., 2005, The estimation of the Cobb Douglas function, Eastern Econom. J. 31(3), 427 – 445.

Glen G. E., 2011, A geometric perspective on the generalized Cobb-Douglas production functions, Appl. Math. Letters 24, 777 – 783.

Mihai A., 2006, Modern Topics in Submanifold Theory, Editura Universităţii din Bucuresti.

Mihai A., Sandu M., 2012, The use of the h-homogeneous production functions in microeconomics. Modelling challenges, Revista Economica Suppl. 1, 465-472.

Mishra S. K., 2010, A brief history of production functions, The IUP Journal of Managerial Economics 8(4), 6 – 34.

Vîlcu A. D., Vîlcu G. E., 2011, On some geometric properties of the generalized CES production functions, Appl. Math. Comp. 218, 124 – 129.