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P-ADIC NUMBERS AND APPLICATIONS IN AND OUTSIDE MATHEMATICS – AN OVERVIEW

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Abstract

The concept of p-adic number was first introduced by Hensel in 1897, but it can be found in some previous works of Kummer. The main motivation for their introduction was the use of some techniques of mathematical analysis in number theory. For example, they play a keyrole in proving Fermat's Last Theorem. The p-adic numbers have important applications in physics (quantum physics, string theory), but in the recent years they have been used in other domains such: computer sciences, cognitive sciences, psychology, sociology, biology and genetics. In this paper we explain the notion of p-adic number and we briefly present some applications with references.

Key words: *p-adic numbers, applications, string theory, quantum physics, biology.*

INTRODUCTION

In this article, I intend to present the notion of p-adic number and the philosophy behind it. I do not intend to show applications in detail in different areas, because it would be time-consuming and very technical, but I give representative references that the interested reader may follow. The paper is written in a way it can be read both by mathematicians and non-mathematicians.

p-Adic numbers were introduced for the first time by Hensel in 1897, but the concept can be found found, without being explicitly named so in some of Kummer's earlier works. The main motivation for their introduction was the use of some mathematical analysis techniques (especially series theory) in number theory. The closer two *p*-adic numbers are, the more their difference is divisible by a greater power of the prime number *p*.

In 1918, Ostrowski proved that any norm on the field of rational numbers $\mathbb{Q}$ is topologically equivalent either with the usual real module, or with the *p*-adic module for a certain prime number *p*. Topologically completing the set $\mathbb{Q}$ in relation to the usual module we obtain $\mathbb{R}$, and in relation to the *p*-adic we obtain $\mathbb{Q}_p$, the field of the numbers.

The complex analysis techniques mentioned above were aimed at local development of an analytical function in a power series. The first attempt came in 1930 with Schobe's PhD thesis, but the most successful was Krasner in the 1950's, inspired by Runge's theorem from the classical analysis of approximation of analytical functions by rational functions, using a simplified method of Weierstrass's for the analytical continuation. Subsequently, in 1961, the study of *p*-analysis triumphed by the works of Tate who used Gröthendieck's ideas, giving a rigid topological structure of the analytic spaces over the *p*-adic fields.

CONSTRUCTION OF REAL NUMBERS

First, I briefly recall the process of construction of real numbers in mathematics. One starts by constructing the set of natural numbers:

$$\mathbb{N} = \{0,1,2,\dots\},$$

then the set of integer numbers:

$$\mathbb{Z} = \{\dots -2,-1,0,1,2,\dots\},$$

then the set of rational numbers:

$$\mathbb{Q} = \left\{\frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{N}^*\right\}$$

Definition 1. The usual *module* (or *absolute value*) on $\mathbb{Q}$ is “the positive part of a number”

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Definition 2. *A Cauchy sequence (or a fundamental sequence)* is a sequence (a_n) with the following property:
For all $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$, such that for any $m, n \in \mathbb{N},\ m, n \geq n_\varepsilon$, one has

$$|a_m - a_n| < \varepsilon$$

One may find Cauchy sequences on $\mathbb{Q}$ that have the limits outside $\mathbb{Q}$ **(**for example with the limit $\sqrt{2}$), and this fact shows that $\mathbb{Q}$ is not a complete space with respect to the usual module, which intuitively means that the set of rational numbers $\mathbb{Q}$ cannot be geometrically represented on a straight line, because there would be gaps. Thus, one has to complete $\mathbb{Q}$ with respect to the usual module, by considering the classes of Cauchy sequences, and obtains the set of real numbers $\mathbb{R}$, which can be intuitively represented on a whole straight line.

CONSTRUCTION OF P-ADIC NUMBERS

I recall the intuitive definition of the notion of *commutative field* in mathematics, which means a set $(K, +, \cdot)$ in which the arithmetic happens in the usual way.

Definition 3. Let *K* be commutative field. We define by *norm* (or *absolute value*) on *K* a function $\|\cdot\|: K \to \mathbb{R}_+$ a function with the properties (Nițu, C.C., 2017):

1) $\|x\| = 0 \Leftrightarrow x = 0$;
2) $\|xy\| = \|x\| \cdot \|y\|$;
3) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in K$.

Definition 4. *The p-adic norm* (or *p-adic module*)

$$|x|_p = \begin{cases} p^{-v_p(x)}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

where $v_p(x)$ represents the *p*-adic exponent.

Example. For $x = \frac{63}{2} = 2^{-1} \cdot 3^2 \cdot 7$, we have $|x|_2 = 2, |x|_3 = 3^2, |x|_7 = \frac{1}{7}$ .and $|x|_p = 1$, for every prime number $p \neq 2,3,7$.

Definition 5. Let *K* be a commutative field. We define by *valuation* on *K* a function $v: K \to \mathbb{R}$ with the properties:

1) $v(0) = \infty$;
2) $v(xy) = v(x) + v(y)$;
3) $v(x + y) \geq \min\{v(x), v(y)\}$,

for all $x, y \in K$.

Definition 6. Let *M* be a non-empty set. By a *distance (or a metric)* on *M* we mean a function $d: M \times M \to \mathbb{R}_+$ whith the properties:

1) Separability:

$$d(x, y) \geq 0, d(x, y) = 0 \Leftrightarrow x = y.$$

2) Symmetry: $d(x, y) = d(y, x)$
3) Triangle's inequality:

$$d(x, y) \leq d(x, z) + d(z, y),$$

for all $x, y \in M$.
The pair (M, d) is called a *metric space.*

Definition 7. Let (M, d) be a metric space. We define the open ball centered at *a* and of radius $\varepsilon > 0$:

$$B(a, \varepsilon) = \{x \in M \mid d(x, a) < \varepsilon\}$$

and the closed ball cantered at *a* and of radius $\varepsilon > 0$:

$$B[a, \varepsilon] = \{x \in M \mid d(x, a) \leq \varepsilon\}$$

Remark 1. These balls are the correspondents of the real open and closed intervals centred at the point a, $(a - \varepsilon, a + \varepsilon)$ and $[a - \varepsilon, a + \varepsilon]$.

Definition 8. A norm on *K* is called *non-archimedian* if $\|x + y\| \leq \max\{\|x\|, \|y\|\}$, for all $x, y \in K$, and *archimedean* otherwise. Also, a distance is called non-archimedean if $d(x, y) \leq \max\{d(x, z), d(y, z)\}$, for all $x, y, z \in K$ and archimedean otherwise. In particular, the distance induced by a the non-archimedean norm is non-archimedean. A non-archimedean metric space is also called an *ultrametric space*.

Remark 2. If $v: K \to \mathbb{R}$ is a valuation on *K* and $c \in (0,1)$, then $\|\cdot\|: K \to \mathbb{R}_+$, defined by $\|x\| = c^{v(x)}$ is a non-archimedian norm on *K*.

Definition 9. Let *X* be a set and τ a family of subsets of X. τ is called a *topology* (from the greek words "topos" (place) and "logos" (study)) if it has the following properties:

1) $\phi, X \in \tau$;
2) $(A_i)_i \in I \in \tau \Rightarrow \bigcup_{i \in I} A_i \in \tau$;
3) $A, B \in \tau \Rightarrow A \cap B \in \tau$.

A set $A \in \tau$ is called an *open* set, and its complement is called a *closed* set. A pair (X, τ) is called a *topological space*. It is easy to show that every metric space has a natural structure of a topological space, the topology being generated by the open balls.
Topological spaces are used to define continuous functions in the most general way.

Definition 10. Let (X, τ_X) and (Y, τ_Y) be two topological spaces. A function $f: X \to Y$ is named *continuous* if for all $U \in \tau_Y$, the preimage $f^{-1}(U) \in \tau_X$.

Definition 11. We say that two metrics (or norms) defined on the same field *K* are *equivalent* if they generate the same topology on *K*.

Theorem 1. (Ostrowski (born in Kiev), 1918)
A nontrivial norm defined on the the field of rational numbers $\mathbb{Q}$ is either equivalent with the usual module, or with the p-adic module, for a certain prime number *p*.

Remark 3. This is a very important theorem which establishes the **philosophy** behind the *p*-adic numbers: **there are only two possible ways in which one phenomenon can be analysed: the real way, or the *p*-adic way**. Therefore, it is natural to investigate *p*-adic mathematical modelling in different areas and to compare them with the models in real numbers.

Definition 12. Two sequences of rational numbers (a_n) and (b_n) are called *equivalent* and we write $(a_n) \sim (a_n)$ if for every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ with the property

$$|a_n - b_n|_p < \varepsilon, \text{for all } n \geq n_\varepsilon .$$

It easy to show that "$\sim$" defined above is an equivalence relation. We denote by $\overline{(a_n)}$ the equivalence class of the sequence (a_n).

Definition 13. *The field of p-adic numbers* is the topological closure of $\mathbb{Q}$ with respect to the *p*-adic norm:

$$\mathbb{Q}_\text{p} = \left\{\overline{(a_n)} \mid (a_n) \subset \mathbb{Q}, \text{Cauchy sequence}\right\}$$

We also denote by $\mathbb{Z}_\text{p} = \left\{x \in \mathbb{Q}_\text{p} \mid \ |x|_p \leq 1\right\}$ *the ring of p-adic integers*.

Proposition 1. If $a = \overline{(a_n)} \in \mathbb{Z}_\text{p}$, then there exists a unique sequence $(c_n), c_n \in \mathbb{N}, 0 \leq c_n < p$, such that

$$a_n = \sum_{k=0}^{n-1} c_k p^k, \ \ for\ all\ n \in \mathbb{N}$$

Thus, one can write

$$a = c_0 + c_1 p + c_2 p^2 + \cdots$$

Furthermore, if $a \in \mathbb{Q}_\text{p} \backslash \mathbb{Z}_\text{p}$, then $\frac{a}{|a|_p} = ap^m \in \mathbb{Z}_\text{p}$ and we can write

$$a = \frac{c_{-m}}{p^m} + \frac{c_{-m+1}}{p^{m-1}} + \cdots + c_0 + c_1 p + c_2 p^2 + \cdots$$

Remark 4. The arithmetic in $\mathbb{Q}_\text{p}$ resembles the one from the *p* base of natural numbers, but the computation is done from "left to write".

Example. If $p = 5$, in $\mathbb{Q}_5$, for

$$a = 2 + p + 4p^2 + \cdots$$
$$b = 3 + 2p + 4p^2 + \cdots$$

then

$$a + b = 4p + 3p^2 + \cdots$$
$$a - b = 4 + 3p + 4p^2 + \cdots$$
$$ab = 1 + 3p + 3p^2 + \cdots$$
$$\frac{a}{b} = 4 + 2p + 4p^2 + \cdots$$

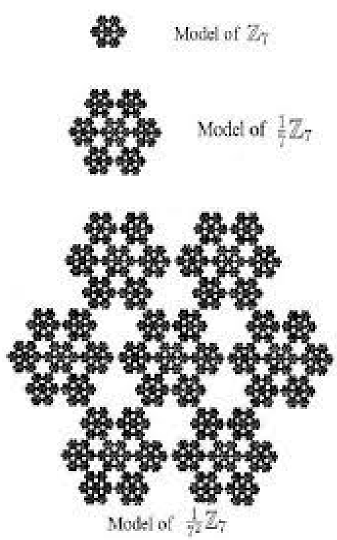

Figure 1. The p-adic balls can be represented in a fractal way (Robert A.M., 2000)

Theorem 2. The algebraic closure $\overline{\mathbb{Q}}_p$ is not a complete metric space with respect to the p-adic module. Let $\mathbb{C}_p = \widetilde{\overline{\mathbb{Q}}_{\mathrm{p}}}$ be the topological closure of $\overline{\mathbb{Q}}_p$. Then $\mathbb{C}_p$ (also named the Tate field), is algebraically closed.

Remark 5. An algebraic closure of a field K is an extension denoted by $\overline{K}$ in which any polynomial from $K[X]$ has all his roots. All algebraic closures of a field are isomorphic.

Similarities between $\mathbb{R}$ and $\mathbb{Q}_p$

- both are fields, completions of $\mathbb{Q}$;
- $\mathbb{Q}$ is dense in each of them;
- they are locally compact spaces;
- they are not algebraically closed;
- we can use analysis techniques that have many similarities.

Differences between $\mathbb{R}$ and $\mathbb{Q}_p$

- $\mathbb{R}$ is an ordered field, the order relation being compatible with algebraic operations "+" and "·";
- $\mathbb{R}$ is archimedean, and $\mathbb{Q}_p$ is non-archimedean;
- $\mathbb{R}$ is a connected topological space, while $\mathbb{Q}_p$ is completely disconnected;
- On $\mathbb{Q}_p$ we cannot clearly define the notions of interval or curve.

General principles of ultrametric calculus

- The strongest wins:
$$|x| > |y| \Rightarrow |x+y| = |x|;$$
- Equilibrum: Every triangle is isoscel (or equilateral):
$$a+b+c=0,\ |c| < |b| \Rightarrow |a| = |b|;$$
- Competition:
$$a_1 + a_2 + \cdots + a_n = 0 \Rightarrow \text{there exist } i \neq j \text{ such that } |a_i| = |a_j| = \max_{k=\overline{1,n}} |a_k|;$$
- A dream came true: (a_n) is a Cauchy sequence $\Leftrightarrow |a_{n+1} - a_n| \to 0$;
- A first-year student's dream: the series $\sum_{n\geq 1} a_n$ is convergent $\Leftrightarrow a_n \to 0$;
- Stability of the absolute value:
$$a_n \to a \Rightarrow there\ exists\ N \in \mathbb{N} \text{ such that } |a_n| = |a|, for\ all\ n \geq N.$$

Elements of *p*-adic topology

- in a non-archimedean field the balls and spheres are open sets and closed sets; any two balls of the same kind are either disjoint or one contains the other. A set that is open and closed it is called a *clopen*;
- any point on a ball can be considered the center of that ball;
- any non-archimedean field K is totally disconnected, that is, the only connected subsets are those of the form $\{a\}$, $a \in K$;
- $\mathbb{Z}$ is dense in $\mathbb{Z}_p$. $\mathbb{Q}$ is dense in $\mathbb{Q}_p$;
- $\mathbb{Z}_p$ is compact and $\mathbb{Q}_p$ is locally compact;
- Any finite extension of $\mathbb{Q}_p$ is locally compact;
- $\mathbb{C}_p$ is not locally compact. In fact, it can be shown that
$$\mathrm{B}[0,1] = \{x \in \mathbb{C}_p \,|\, |x|_p \leq 1\}$$
is not compact.

APPLICATIONS

When they were first introduced, *p*-adic numbers where considered an exotic part of pure mathematics, without practical applications.

Soon after that, their property of being closer when their difference is divisible by a greater power of *p*, showed they were very useful in encoding properties of modulo *p* congruences and allowed the use of new analytical methods in number theory (for applications to mathematics see Koblitz N., 1977; Manin Yu & Panchishkin A.A., 2007). Thus, they turned out to have important applications in number theory, for example in the proof of Fermat's Last Theorem (Wiles, A.J., 1995).

Remark 6. Fermat's Last Theorem (FLT), was conjecture by the French mathematician Pierre de Fermat in 1637 and states that the equation
$$x^n + y^n = z^n$$
has no strictly positive natural solutions if $n \geq 3$.

The efforts of proving it by generations of mathematicians led to the development of modern algebra.

Since the 1960's physicist became interested in creating new models of space-time for the description of the very small Plank distances. There are evidences that the standard model based on real numbers is not correct, and that the p-adic numbers may give a better description due to some of their properties, such the fact that they are not ordered.

Thus, p-adics are an important tool for mathematical modelling in string theory and quantum mechanics (Rozikov, U.A., 2013; Araf'eva, L.Ya. et al., 1991; Freund P.G.O. & Witten E, 1987; Khamraev M. et al., 2004; Vladimirov V.S et al., 1994). This interest in physics gave rise to the development of new mathematical branches such as: theory of p-adic distributions (Albeverio, S. et al., 2010; Khrennikov, A.Yu., 1994) p-adic differential equations (Khrennikov, A.Yu., 1990; Vladimirov, V.S. et al., 1990), p-adic probability theory (Khrennikov, A.Yu., 1994; Vladimirov, V.S. et al., 1990) and p-adic spectral theory (Albeverio, S. & Khrennikov, A.Yu, 1996; Albeverio, S. et al., 1997; Khrennikov, A.Yu., 1997).

The representation of p-adic numbers as a sequence of digits allowed the use of this number system for encoding information. In particular, they can be used in cognitive sciences, psychology and sociology.

Such models are based on p-adic dynamical systems (Albeverio, S. et al., 2013; Albeverio, S. et al., 1999; Albeverio, S.A. et al., 1998; Anashin, V. & Khrennikov, A., 2009; Silverman, J., 2007).

For example, the human process of thinking can be modelled by a dynamical system that works with ideas or sets of ideas, of the form

$$x_{n+1} = f(x_n), \qquad x_n \epsilon X$$

where $X = \mathbb{Z}_p$ is the configuration space of dynamical system, the "space of ideas", and f is an analytic function on X (a function that can be developed locally at the point x_0 as a Taylor series: $f(x) = \sum_{n\geq 0} a_n (x - x_0)^n$).

There also exist effective results in areas such: computer science (straight line programs, cryptography, automata theory, formal languages), numerical analysis and simulations (Anashin, V. & Khrennikov, A, 2009; Anashin, V.S., 2011; Anashin, V., 2010; Anashin, V., 2012; Fan, A.H. & Liao, L.M., 2011; Kingsbery, J. 2009; Kingsbery, J., 2011; Lin, D. et al., 2012; Pin, J.E., 2009; Shi, T. et al., 2012).

An automaton is a relatively self-operating machine, or control mechanism designed to automatically follow a sequence of operations, or respond to predetermined instructions. For example, a finitness criteria for automata presented in an article mentioned above states that: Given a Lipschitz function $f: \mathbb{Z}_p \to \mathbb{Z}_p$ represented by the van der Put series

$$\sum_{n\geq 0} b_n p^{[\log_p n]} X(m, x)$$

then the function f is the automaton function of a finite automaton if and only if the following conditions are satisfied:

i) $B_f = \{b_n | \ n \geq 0\}$ is a finite subset of $\mathbb{Q} \cap \mathbb{Z}_\mathrm{p}$;

ii) the p-kernel of the sequence (b_n) is finite.

Remark 7. A function $f: (X, d_1) \to (Y, d_2)$ defined between two metric space is called Lipschitz if there exists $\lambda > 0$ such that

$$d_1\big(f(x), f(y)\big) \leq \lambda \, d_2(x, y) \, .$$

Remark 8. $X(n,x) = \begin{cases} 1, & if \ |x - n| \leq p^{-m} \\ 0, & otherwise \end{cases}$

Remark 9. The p-kernel of the sequence $b = (b_n)$ is the set $\ker(b)$ of all subsets $(b_{kp^m+t})_{\ k\geq 0}$, where $0 \leq t < p^m$ is fixed.

Remark 10. $[\log_p n]$ (the lower integer part) represents the number of digits of the representation of n in the p base.

Also, some applications of p-adic numbers to biology and genetics have been proposed (see Albeverio, S. et al., 2013;
Albeverio, S., 1999; Albeverio S., 1998; Khrennikov, A. Yu. Et al., 1997). It is considered that p-adic numbers can be used to model biological systems and phenomena with a hierarchical structure.

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