Scientific Papers. Series E. Land Reclamation, Earth Observation & Surveying, Environmental Engineering. Vol. XII, 2023
Print ISSN 2285-6064, CD-ROM ISSN 2285-6072, Online ISSN 2393-5138, ISSN-L 2285-6064

DIFFERENTIALS AND APPLICATIONS TO FUNCTION APPROXIMATIONS

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Abstract

In mathematics, the term "differential" refers to several related notions derived from the early days of mathematical analysis, which became rigorous later, such as infinitesimal differences and the derivatives of functions. The notion is used in various areas of mathematics such as algebraic geometry, algebraic topology, calculus, differential geometry etc. The term differential is used non rigorously in calculus referring to an infinitely small ("infinitesimal") variation change in a quantity. For example, if one considers x as a variable, then a "bigger" change in the value of x is often denoted by Δx. The differential dx is an infinitesimal change of the variable x. The concept of an infinitely small or infinitely slow change is very useful, and there are several of mathematical tools to make it precise. Using calculus, it is possible to relate the infinitesimal changes of several variables to each other using function derivatives. In this article we present the notions of Gateaux and Fréchet differentials of a multivariable function with their properties, geometric interpretation and applications to function approximations.

Key words*: differential, multivariable function, function approximation.*

INTRODUCTION

The derivative of a one variable function

Definition 1. A function $f: I \to \mathbb{R}$ is derivable at a point $a \in I$ if the following limit exists:

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = f'(a) \in \mathbb{R} \qquad (1)$$

In this case, we also denote $f'(a) = \frac{df}{dx}(a)$.

Remark 1. If f is derivable at the point $a \in I$ then f is continuous at a, because:

$$\lim_{h\to 0} f(a+h) - f(a) = 0$$

implies that:

$$\lim_{x\to a} f(x) = f(a).$$

Geometric interpretation of the first order derivative of a single variable real function is presentet in Figure 1.

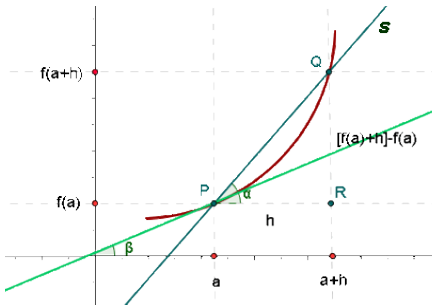

Figure 1. Geometric interpretation of the first order derivative of a single variable real function (https://www.superprof.co.uk/resources/academic/maths/calculus/derivatives/physical-interpretation-of-the-derivative.html)

The graph of f has a tangent at the point $(a, f(a))$

$$y - f(a) = f'(a)(x-a) \qquad (2)$$

Also we can define the linear application $T: \mathbb{R} \to \mathbb{R}$, $T(h) = f'(a)h$, which in a vicinity of the point a represents an approximation of the $f(a+h) - f(a)$, because:

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}-f'(a)=0$$

implies that:

$$\lim_{h\to 0}\frac{f(a+h)-f(a)-T(h)}{h}=0$$

Thus

$$\lim_{h\to 0} f(a+h)-f(a)-T(h)=0,$$

which implies that:

$$f(a+h)-f(a)=T(h)+o(h) \qquad (3)$$

where: $o(h)$ is a function with the property

$$\lim_{h\to 0} o(h)=0$$

Remark 2

$$f(a+h)-f(a)\approx T(h), \text{ when } h\to 0$$

Remark 3. The notation $\Delta x := h$ is mainly used in practice for approximations, being asimilated with an increase of x. With this one can write:

$$f(a+\Delta x)-f(a)\approx f'(a)\Delta x, \text{ when } \Delta x\to 0 \qquad (4)$$

And, even further

$$\Delta f(a)\approx f'(a)\Delta x, \quad \text{when } \Delta x\to 0 \qquad (5)$$

which means that:

$$f(x)\approx f(a)+f'(a)(x-a), \text{ when } x\to a \qquad (6)$$

MATERIALS AND METHODS

In this paper we will consider $x=(x_1,x_2,\dots,x_n)$ and $f=f(x)=f(x_1,x_2,\dots,x_n)$

Definition 2 (Colojoară, 1983; Gateux differentiation) Let $U\subset\mathbb{R}^n$ be an open set, $f: U\to\mathbb{R}$ a function and $u\in U$. We say that *f is derivable on the direction a* if the following limit exists:

$$\lim_{h\to 0}\frac{f(x+hu)-f(x)}{h} := \frac{\partial f}{\partial u}(x) \qquad (7)$$

If $u=e_i=(0,0,\dots 0,1,0\dots 0)$, where 1 lies on the *i*-th position, then we denote for simplicity

$$\frac{\partial f}{\partial x_i} := \frac{\partial f}{\partial e_i}, \qquad i=\overline{1,n} \qquad (8)$$

Definition 2. We say that $T:\mathbb{R}^n\to\mathbb{R}^m$ is a *linear application* (or a *functional, or a morphism* of $\mathbb{R}$ vectorial spaces) if it has the property:

$$T(\alpha x+\beta y)=\alpha f(x)+\beta f(y), \quad \forall x,y\in\mathbb{R}^n, \alpha,\beta\in\mathbb{R} \qquad (9)$$

We denote by $L(\mathbb{R}^n,\mathbb{R}^m)$ the space of all such functionals.

Definition 3. (Nicolescu, et al., 1971; Fréchet differentiation) Let $U\subset\mathbb{R}^n$ be an open set, $f: U\to\mathbb{R}$ a function and $a\in U$. We say that *f is differentiable at the point a* if there exists a continuous functional $T\in L(\mathbb{R}^n,\mathbb{R}^m)$ such that

$$\lim_{h\to 0}\frac{f(x+ha)-f(x)-hT(a)}{h}=0 \qquad (10)$$

In case of existence, we denote

$$T(a):=df(a) \qquad (11)$$

Remark 4. If $T\in L(\mathbb{R}^n,\mathbb{R}^m)$, then:

$$dT(a)=T(a) \qquad (12)$$

because:

$$\lim_{h\to 0}\frac{T(x+ha)-T(x)-T(a)}{h} = \lim_{h\to 0}\frac{T(x)+hT(a)-T(x)-T(a)}{h}$$

Remark 5. If *f* is differentiable then there exists $\varphi: U\to\mathbb{R}^m$ such that

$$f(x)=f(a)+T(x-a)+\|x-a\|\varphi(x), \quad \forall x\in U \qquad (13)$$

and

$$\lim_{x\to a}\varphi(x)=0 \qquad (14)$$

Lema 1. In the conditions from above the application *T* is uniquely determined.

Proof. We assume the existence of another linear application $T\in L(\mathbb{R}^n,\mathbb{R}^m)$ such that

$$f(x)=f(a)+T_1(x-a)+\|x-a\|\psi(x), \forall x\in U$$

where

$$\lim_{x\to a}\psi(x)=0.$$

By denoting

$$T - T_1 = T_2 \in L(\mathbb{R}^n, \mathbb{R}^m),$$

and

$$\alpha(x) = \varphi(x) - \psi(x)$$

we have

$$T_2(x-a) = \|x-a\|\alpha(x), \qquad \forall x \in U$$

where:

$$\lim_{x \to a} \alpha(x) = 0.$$

For $h \in \mathbb{R}^n$, fixed, and $t > 0$ sufficiently small, taking $x = a + th$ we get

$$T_2(th) = \|th\|\ \alpha(a+th)$$

And due to liniarity one has

$$T_2(h) = \|h\|\ \alpha(a+th)$$

Thus, for $t \to 0$ it results

$$T_2(h) = 0, \forall\ h \in \mathbb{R}^n$$

Therefore

$$T_2 = 0,$$

which shows that

$$T = T_1$$

and the unicity is proved.

Theorem 1. (Stănășilă, O. (1981), Boboc, N. (1999), the connexion between Gateaux and Fréchet differentiation) Let $f: U \to \mathbb{R}$ be a function defined on an open set $\subset \mathbb{R}^n$.

i) If *f* is differentiable at a point $a \in U$ then *f* is continuous in *a.* Furthermore, there exists $\frac{df}{ds}(a)$ for every versor $s \in \mathbb{R}^n$(i.e. vector with the norm equal to 1). In particular, there exist all the first order partial derivatives

$$\frac{df}{ds}(a) = df(a)(s) \qquad (15)$$

and

$$\frac{\partial f}{\partial x_i}(a) = df(a)(e_i), i = \overline{1,n} \qquad (16)$$

where: $E = \{e_i |\ i = \overline{1,n}\}$ is the canonic base of $\mathbb{R}^n$.

ii) If $f \in C^1(U)$ (i.e. the space of derivable functions defined on *U,* with all the partial derivatives continuous) then *f* is differentiable on *U.* Particularly, every elementary function is differentiable.

Proof.

i) On one hand, for any convergent sequence $(x_n) \subset U, x_n \to a$, we have

$$f(x_n) = f(a) + df(a)(x_n - a) + \|x_n - a\|\varphi(x_n), \forall n \in \mathbb{N}$$

Therefore, if $n \to \infty$, it results

$$f(x_n) \to f(a).$$

Thus, *f* is continuous in *a*.

On the other hand, choosing $r > 0$ with the property that $B(a,r) \subset U$, then for any $t \neq 0, |t| < r$, one has $d(a+ts, a) < r$, so

$$a + ts \in U,\ \forall\ k = \overline{1,n}.$$

Furthermore

$$\frac{f(a+ts) - f(a)}{t} = \frac{df(a)(ts) + \|ts\|\varphi(a+ts)}{t} = df(a)(s) + \frac{|t|}{t}\varphi(a+ts)$$

Thus,

$$\lim_{t \to 0} \frac{f(a+ts) - f(a)}{t} = df(a)(s)$$

which means that

$$\frac{df}{ds}(a) = df(a)(s), \qquad for\ s = e_i, \quad i = \overline{1,n}$$

As a result:

$$\frac{\partial f}{\partial x_i}(a) = df(a)(e_i), i = \overline{1,n}$$

ii) One has

$$\begin{aligned} f(x) - f(a) = &[f(x_1, x_2, \dots, x_n) - f(a_1, x_2, x_3, \dots, x_n)] \\ &+ [f(a_1, x_2, x_3, \dots, x_n) - f(a_1, a_2, x_3, \dots, x_n)] + \cdots \\ &+ [f(a_1, a_2, \dots a_{n-1}, x_n) - f(a_1, a_2, a_3, \dots, a_n)] \\ &= (x_1 - a_1)\frac{\partial f}{\partial x_1}(\xi_1, x_2, x_3, \dots, x_n) \\ &+ (x_2 - a_2)\frac{\partial f}{\partial x_2}(x_1, \xi_2, x_3, \dots, x_n) \\ &+ \cdots \\ &+ (x_n - a_n)\frac{\partial f}{\partial x_n}(x_1, x_2, x_3, \dots, \xi_n) \end{aligned}$$

(we applied Lagrange's theorem for finite increases for each term, finding ξ_i between a_i and $x_i, i = \overline{1,n}$).

Now, we define the liniar application

$$T: \mathbb{R}^n \to \mathbb{R},$$

$$T(x) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a) x_i$$

From here

$$T(x-a) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a)$$

thus,

$$\lim_{x\to a}\frac{f(x)-f(a)-T(x-a)}{\|x-a\|}$$
$$=\lim_{x\to a}\frac{(x_1-a_1)\left[\frac{\partial f}{\partial x_1}(\xi_1,x_2,x_3,\dots,x_n)-\frac{\partial f}{\partial x_1}(a)\right]}{\|x-a\|}$$
$$+\lim_{x\to a}\frac{(x_2-a_2)\left[\frac{\partial f}{\partial x_2}(a_1,\xi_2,x_3,\dots,x_n)-\frac{\partial f}{\partial x_2}(a)\right]}{\|x-a\|}+\cdots$$
$$+\lim_{x\to a}\frac{(x_n-a_n)\left[\frac{\partial f}{\partial x_n}(a_1,x_2,x_3,\dots,a_{n-1},\xi_n)-\frac{\partial f}{\partial x_n}(a)\right]}{\|x-a\|}$$

Every limit from above is equal to 0, as the quotients

$$\left|\frac{x_i-a_i}{\|x-a\|}\right|\le 1,\qquad i=\overline{1,n}$$

and because $f\in C^1$, its partial derivatives are continuous and the straight brackets tend to 0, $x\to a$ (which means that $x_i\to a_i,$, $i=\overline{1,n}$).
In conclusion,

$$\lim_{x\to a}\frac{f(x)-f(a)-T(x-a)}{\|x-a\|}=0$$

which shows that f is differentiable at any point $a\in U$.

Theorem 2. (The formula for computing the differential) Let $f: U\to\mathbb{R}$ be a function defined on an open set $U\subset\mathbb{R}^n$, differentiable at a point $a\in U$. Then the following formula takes place:

$$df(a)=\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(a)pr_i \tag{17}$$

(equality of liniar applications)
Proof.
Two liniar applications are equal if and only if they coincide on the canonic base $E=\{e_i|\, i=\overline{1,n}\}$. Applying this property, the desired relation is proven from the fact that

$$pr_i(e_j)=\delta_{ij}$$

where

$$\delta_{ij}=\begin{cases}1, & if\ i=j\\ 0, & if\ i\ne j\end{cases}$$

is the Kronecker symbol.
Thus, for any $h=(h_1,h_2,\dots,h_n)\in\mathbb{R}^n$ one can write

$$df(a)(h_1,h_2,\dots,h_n)=\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(a)h_i \tag{18}$$

We can write this relation more conveniently in the following way: we observe that the projections $pr_i:\mathbb{R}^n\to\mathbb{R}$, defined by

$$pr_i(x)=x_i$$

are liniar applications, so

$$d(pr_i)(a)=pr_i.$$

Thus

$$dx_i(a)=pr_i$$

And the desired formula can be written:

$$df(a)=\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(a)dx_i(a) \tag{19}$$

Remark 6. The function $f:\mathbb{R}^2\to\mathbb{R}$,

$$f(x,y)=\begin{cases}\frac{x^2y}{x^6+y^2}, & if\ (x,y)\ne(0,0)\\ 0, & if\ (x,y)=(0,0)\end{cases} \tag{20}$$

is not continuous in (0,0), because

$$\lim_{\substack{x\to 0^+\\ y=x^3}}f(x,y)=\lim_{x\to 0^+}\frac{1}{x}=+\infty$$

but it has partial derivatives at (0,0), as

$$\frac{\partial f}{\partial x}(0,0)=\lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t}=0$$

and

$$\frac{\partial f}{\partial y}(0,0)=\lim_{t\to 0}\frac{f(0,t)-f(0,0)}{t}=0$$

Remark 7. In practice, one may denote the increase of the variables x_i , resp. of the function f at the point *a,* with Δx_i, resp. $\Delta f(a)$ and one may write:

$$\Delta f(a)\approx\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(a)\Delta x_i\ ,\text{when } \Delta x_i\to 0,\quad i=\overline{1,n} \tag{21}$$

which means that

$$f(x) \approx f(a) + df(a)(x-a) \text{ , when } x \to a \tag{22}$$

Remark 8.

$$|\Delta f(a)| \approx \left|\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)\Delta x_i\right|$$

and using module inequality we obtain

$$|\Delta f(a)| \leq \sum_{i=1}^{n} \left|\frac{\partial f}{\partial x_i}(a)\right| |\Delta x_i| \tag{23}$$

when $\Delta x_i \to 0$

Remark 9. The vector

$$grad_a f = \left(\frac{\partial f}{\partial x_1}(a), \frac{\partial f}{\partial x_2}(a), \ldots, \frac{\partial f}{\partial x_n}(a)\right) \in \mathbb{R}^n \tag{24}$$

is called the *gradient* of f at the point a and the set

$$T_a f = \{x \in \mathbb{R}^n |\, df(a)(x) = 0\} \tag{25}$$

is called the *hyperplane tangent* at a at the hypersurface of equation

$$f(x) = f(a)$$

Geometric interpetation of the differential is presentet in Figure 2.

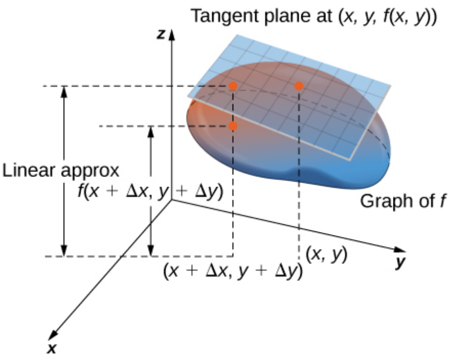

Figure 2. Geometric interpretation of the differential of multivariable function (https://math.libretexts.org/Bookshelves/Calculus/Map%3A_University_Calculus_(Hass_et_al)/13%3A_Partial_Derivatives/13.6%3A_Tangent_Planes_and_Differentials)

Exemple. $f: \mathbb{R}^3 \to \mathbb{R}, f(x,y,z) = x^3 + xyz$

The differential of f at the current point is:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = (3x^2 + yz)dx + xzdy + xydz$$

In particular, for every
$a = (a_1, a_2, a_3)$ and $h = (h_1, h_2, h_3) \in \mathbb{R}^3$
one has

$$df(a)(h_1, h_2, h_3) = (3a_1^2 + a_2a_3)h_1 + a_1a_3h_2 + \mathrm{a_1a_2h_3}$$

Also the increase of f is:

$$\Delta f \approx (3x^2 + yz)\Delta x + xz\Delta y + xy\Delta z$$

APPLICATIONS

For more details and applications see Martin, O. (2008), Şerban S et al., (2015) and Tudor, H. (2008).

Practical example 1: Aproximate cos 61° using differentials.
Solution:
Considering the function $y = \cos x$

$$\Rightarrow dy = -sinx\ dx$$
$$\Rightarrow \Delta y \approx -sinx\ \Delta x$$
$$x = 60^\circ = \frac{\pi}{3}\ rad$$
$$\Delta x = 61^\circ - 60^\circ = 1^\circ = \frac{\pi}{180}\ rad$$
$$\cos 61^\circ = \cos 60^\circ + \Delta y \approx \frac{1}{2} - \frac{\sqrt{3}\pi}{360} \approx 0.484$$

Practical example 2: Consider a sphere with the radius R=10 m. Determine the aproximative increase of the volume, if the radius increases with $\Delta R = 1\ m$.
Solution:

$$V = \frac{4\pi R^3}{3} dV = 4 \Rightarrow \pi R^2 dR \Rightarrow \Delta V \approx 4\pi R^2 \Delta R \approx 1256\ m^3$$

Practical example 3: The radius and the height of a right-circular cylinder are measured with an error of at most 2%. Approximate the maximum percentage error in the volume.
Solution:

$$V = \pi r^2 h \Rightarrow \begin{cases} \dfrac{\partial V}{\partial r} = 2\pi r h \\ \dfrac{\partial V}{\partial h} = \pi r^2 \end{cases}$$

$$dV = \frac{\partial V}{\partial r}\, dr + \frac{\partial V}{\partial h} dh$$
$$\Rightarrow dV = 2\pi rh\, dr + \pi r^2 dh$$
$$\Rightarrow\ \Delta V \approx 2\pi rh\, \Delta r + \pi r^2 \Delta h$$
$$\Rightarrow\ \frac{\Delta V}{V}\ \approx 2\ \frac{\Delta r}{r} + \frac{\Delta h}{h} = 6\%$$

Practical example 4: The lengths of a rectangle are measured with an error 1%. Estimate the maximum percentage error in the area.

Solution:
$A = L \cdot l$
$$\Rightarrow \begin{cases} \frac{\partial A}{\partial L} = l \\ \frac{\partial A}{\partial l} = L \end{cases} \Rightarrow dA = \frac{\partial A}{\partial L} dL + \frac{\partial A}{\partial l} dl$$
$$\Rightarrow\ \Delta A \approx l\, \Delta L + L \Delta l$$
$$\Rightarrow\ \frac{\Delta A}{A}\ \approx\ \frac{\Delta L}{L} + \frac{\Delta l}{l} = 1\% + 1\% = 2\%$$

Practical example 5: The lengths of a triangle are measured with an error 1%, and the angles are measured with a precision of 2%. Estimate the maximum percentage error in the area.
Solution:
Considering a triangle MPQ the area is given by the formula: $A = \frac{mq\ sinP}{2}$
(we measure the sides $m = PQ, q = MP$ and the angle P, where P is the widest acute angle)

$$\Rightarrow \begin{cases} \frac{\partial A}{\partial m} = \frac{q\ sinP}{2} \\ \frac{\partial A}{\partial q} = \frac{m\ sinP}{2} \\ \frac{\partial A}{\partial P} = \frac{mq\ cosP}{2} \end{cases}$$
$$\Rightarrow dA = \frac{\partial A}{\partial m} dm + \frac{\partial A}{\partial q} dq + \frac{\partial A}{\partial P} dP$$
$$\Rightarrow \Delta\ A \approx \frac{\partial A}{\partial m} \Delta m + \frac{\partial A}{\partial q} \Delta q + \frac{\partial A}{\partial P} \Delta P$$
$$\Rightarrow\ \frac{\Delta A}{A}\ \approx\ \frac{\Delta m}{m} + \frac{\Delta q}{q} + Pctg(P) \frac{\Delta P}{P}$$
$$= 0{,}02(1 + Pctg(P)) \leq 0.04$$

Practical example 6: $\rho = \frac{m}{V}$,
$\Delta m =$ the error on the scale,
$\Delta V =$ volume′s measurement error, $\Delta\rho =?$

Solution:
The differential of ρ is:
$$d\rho = \frac{\partial \rho}{\partial m} dm + \frac{\partial \rho}{\partial V} dV$$
$$d\rho = \frac{1}{V} dm - \frac{m}{V^2} dV$$
The increase of ρ is:
$$\Delta\rho \approx \frac{1}{V} \Delta m - \frac{m}{V^2} \Delta V$$
If needed:
$$|\Delta\rho| \leq \frac{1}{V} |\Delta m| + \frac{m}{V^2} |\Delta V|$$

CONCLUSIONS

Differentials are a powerful and useful instrument to study function approximations. This technique is universal and can be applied to any formula that involves elementary mathematical functions, and for any scientific field, from pure mathematics to engineering. Differentials, in general, have wide applications in science: deduction of different formulas and equations, motion description, calculation of profit and losses in economics, differential equations and partial differential equations, mathematical modelling etc.

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