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# A REFINEMENT OF THE SECOND CRITERIA OF COMPARISON FOR THE COVERGENCE OF SERIES OR IMPROPER INTEGRALS

**Cosmin-Constantin NIȚU**

University of Agronomic Sciences and Veterinary Medicine of Bucharest,
59 Marasti Blvd, District 1, Bucharest, Romania

Corresponding author email: cosmin.nitu@fifim.ro

***Abstract***

*The second criterion of comparison for the convergence of series or improper integrals, also called the limit comparison test, is a little bit unnatural and difficult for students when it comes to finding the comparison term. In this paper we give a refinement of this criteria via equivalents, which is more natural. We say that two sequences or two functions are equivalent if their quotient tends to 1 at some point. When two sequences or two integrable functions are equivalent, then their associated series or integrals have the same nature. We also present different applications of this method for both series and improper integrals.*

***Key words***: *equivalent, limit comparison test, improper integrals, series.*

## INTRODUCTION

### The notion of series

In mathematics, a series represents a sum with an infinite number of terms. We are interested in such a sum can be correctly defined, that is if it has a limit, and, if possible, to calculate its sum. A series with a finite sum is called *convergent*.

***Definition 1***. Let $(x_n)_{n\geq 1}$ be a positive sequence of real numbers (Colojoară, 1983). A *series* with positive terms is a sum with an infinite number of terms of the form:

$$x_1 + x_2 + \cdots + x_n + \cdots$$

which can be written in a more compact way:

$$\sum_{n=1}^{\infty} x_n \ or \sum_{n\geq 1} x_n$$

***Remark 1***.

$$\sum_{n\geq 1} x_n \neq \lim_{n\to\infty} x_n$$

***Remark 2***. $x_n$ is called the *general term* of the series.

*Observation.* A series may start from any natural index, for example:

$$x_k + x_{k+1} + \cdots + x_n + \cdots = \sum_{n=k}^{\infty} x_n = \sum_{n\geq k} x_n$$

We will write in a simpler manner $\sum x_n$ for series or $(x_n)$ for sequences when the starting index is not important.

### The convergence of a series with positive terms

We denote $S_n = x_k + x_2 + \cdots + x_n$ the partial sum of order *n* of a series:

$$\sum_{n\geq k} x_n$$

***Remark 3***. A sequence of real numbers $(a_n)$ is called *convergent* to a number $L \in \mathbb{R}$ if:

$$\lim_{n\to\infty} x_n = L$$

***Definition 2***. The series $\sum x_n$ is convergent if and only if the sequence $(S_n)$ e convergent, more exactly:

$$\sum x_n = a \overset{def}{\Longleftrightarrow} \lim_{n\to\infty} S_n = a$$

where:

- *a* is called the *sum* of the series.

***Remark 4***. The convergence of a series is not affected by its starting term.

A theorem of Weierstrass states that a monotonous sequence (i.e. increasing or decreasing) and bounded is convergent.

Therefore, *the nature* (*convergence*) of a series with positive terms is:

$$\begin{cases} convergent\ (if\ bounded): \sum x_n = a \in \mathbb{R} \\ divergent\ (if\ unbounded): \sum x_n = \infty \end{cases}$$

***Remark 5***. In general, it is easier to study the convergence of a series, because there exist certain criteria, than to calculate its sum.

**Classical series**

There are two important series that can be used in the comparison tests:

1) *The generalized harmonic series*:

$$\sum_{n\geq 1} \frac{1}{n^\alpha} = 1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \cdots$$

$$is \begin{cases} conv, & \alpha > 1 \\ div, & \alpha \leq 1 \end{cases}$$

*Example 1*.

$$\sum_{n\geq 1} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$$

is convergent because $\alpha = 2 > 1$.

***Remark 6***. The fact that its sum is $\frac{\pi^2}{6}$ can be proved with advanced techniques of trigonometry and complex analysis. This shows that a series' sum is sometimes hard or even impossible to calculate.

*Example 2*.

$$\sum_{n\geq 1} \frac{1}{n} = \sum_{n\geq 1} \frac{1}{n^1} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = \infty$$

is divergent because $\alpha = 1 \leq 1$.

***Remark 7.*** The divergence of this series is a little bit unnatural for a beginner, as its general term becomes smaller and smaller $\left(\frac{1}{n} \to 0\right)$.

2) *The geometric series:*

$$\sum_{n\geq 0} a^n = 1 + a + a^2 + \cdots$$

$$is \begin{cases} conv = \frac{1}{1-a}, & for\ a < 1 \\ div, & for\ a \geq 1 \end{cases}$$

*Example*:

$$\sum_{n\geq 0} \frac{1}{3^n} = \sum_{n\geq 0} \left(\frac{1}{3}\right)^n$$

is convergent because $a = \frac{1}{3} < 1$, and its sum is:

$$\sum_{n\geq 0} \left(\frac{1}{3}\right)^n = \frac{1}{1-a} = \frac{1}{1-\frac{1}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

**Improper integrals**

The classical Riemann integral of an integrable function $f: [a, b] \to \mathbb{R}$ can be called a *proper integral.*

$$\int_a^b f(x)dx = \int_{[a,b]} f(x)dx$$

***Definition 3***. Let $a \in \mathbb{R}$. We define *improper integrals* of the form:

$$\int_I f(x)dx$$

where:

- $I \subset \mathbb{R}$ is an interval and $f(x)$ is a function Riemann integrable.

1) $I = [a, \infty)$

$$\int_{[a,\infty)} f(x)dx = \int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

2) $I = (-\infty, a]$

$$\int_{(-\infty,a]} f(x)dx = \int_{-\infty}^a f(x)dx = \lim_{t\to-\infty} \int_t^a f(x)dx$$

3) $I = \mathbb{R}$

$$\int_{\mathbb{R}} f(x)dx = \int_{-\infty}^\infty f(x)dx$$

$$= \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$$

where:

- $a$ can be any real number, most commonly 0 or 1.

***Remark 8***. Notations for lateral limits of a function in one variable at a point $a \in \mathbb{R}$:

*The limit to the left*:

$$l_s = f_s(a) = \lim_{\substack{x\to a \\ x<a}} f(x) = f(a-0) = f(a-)$$

*The limit to the right*:

$$l_d = f_d(a) = \lim_{\substack{x\to a \\ x>a}} f(x) = f(a+0) = f(a+)$$

4) $I = [a, b)$

$$\int_{[a,b)} f(x)dx = \int_a^{b-} f(x)dx = \lim_{\substack{t\to b \\ t<b}} \int_a^t f(x)dx$$

5) $I = (a, b]$

$$\int_{(a,b]} f(x)dx = \int_{a+}^{b} f(x)dx = \lim_{\substack{t\to a\\ t>a}} \int_{t}^{b} f(x)dx$$

6) $I = (a, b)$

$$\int_{(a,b)} f(x)dx = \int_{a+}^{b-} f(x)dx$$

$$= \int_{a+}^{c} f(x)dx + \int_{c}^{b-} f(x)dx$$

where:

- $c \in (a, b)$ can be chosen arbitrary, most commonly $c = \frac{a+b}{2}$.

7) $I = (a, \infty)$

$$\int_{(a,\infty)} f(x)dx = \int_{a+}^{\infty} f(x)dx$$

$$= \int_{a+}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$$

where:

- $c \in (a, \infty)$ can chose arbitrary.

8) $I = (-\infty, a)$

$$\int_{(-\infty,a)} f(x)dx = \int_{-\infty}^{a-} f(x)dx$$

$$= \int_{-\infty}^{c} f(x)dx + \int_{c}^{-\infty} f(x)dx$$

where:

$- c \in (-\infty, a)$ can chosen arbitrary.

***Definition 4***. An improper integral is called *convergent* if it is finite and *divergent* if it is infinite or does not exist.

There are many connections between series and improper integrals: the terminology for convergence, similar criteria, the integral criterion for series, etc.

***Theorem 1***. The integral criterion for positive series (Boboc, 1999; Miculescu, 2010).

Let $f: [a, \infty) \to \mathbb{R}_+$ be a positive and decreasing function ($i.e.\ x \le y \Rightarrow f(x) \ge f(y)$). Then:

$$\sum_{n\ge a} f(n) \sim \int_{a}^{\infty} f(x)dx$$

The series has the same nature with the integral.

***Remark 9***. The nature of the integral $\int_a^\infty f(x)dx$ doesn't depend on the number *a*.

## MATERIALS AND METHODS

When studying the convergence of a series or improper integral, the classical comparison test with inequalities is sometimes difficult to apply and very sensitive to signs. There is a more practical second comparison test that uses limits.

***Theorem 1.*** Limit comparison criterion for series (Boboc, 1999; Goleț et al., 2014).

If $\lim_{n\to\infty} \frac{x_n}{y_n} = l$ then if:

$$a)\ l \in (0, \infty) \Rightarrow \sum x_n \sim \sum y_n$$

The two series have the same nature.

$$b)\ l = 0 \Rightarrow \begin{cases} a)\ \sum y_n\ conv \Rightarrow \sum x_n\ conv \\ b)\ \sum x_n\ div \Rightarrow \sum y_n\ div \end{cases}$$

$$c)\ l = \infty \Rightarrow \begin{cases} a)\ \sum x_n\ conv \Rightarrow \sum y_n\ conv \\ b)\ \sum y_n\ div \Rightarrow \sum x_n\ div \end{cases}$$

***Remark 10***. In this form, the limit comparison criterion may by difficult to apply, beacause the comparison therm $y_n$ may by difficult and artificial to find. Thus, we propose a more natural way to find the comparison therm, using the language of equivalents.

***Definition 5***.

We say that two sequences $(x_n)$ and $(y_n)$ are *equivalent* and we write:

$$x_n \sim y_n \overset{def}{\Longleftrightarrow} \lim_{n\to\infty} \frac{x_n}{y_n} = 1$$

(Stănășilă, 1981; Martin, 2008).

***Remark 11***. Any expression is equivalent to its dominant term (if it has one!).

For example, $3n^2 - n + 5 \sim 3n^2$, because by the "degree rule" one has

$$\frac{3n^2-n+5}{3n^2} \to \frac{3}{3} = 1$$

***Remark 12***. The order of the elementary sequences at infinity ($n \to \infty$):

$$\ln n \ll n^k \ll a^n \ll n! \ll n^n$$

where:

$k > 0, a > 1$.

By "$x \ll y$" we mean that *x* is "a lot smaller" than *y*, which imply fundamental limits such:

$\lim\limits_{n\to\infty}\frac{\ln n}{n^k}=0,\ \lim\limits_{n\to\infty}\frac{n^k}{a^n}=0,\ \lim\limits_{n\to\infty}\frac{a^n}{\ln n}=\infty$ etc.

***Definition 6***. Let $f,g:\mathbb{R}\to\mathbb{R}$. We $f(x)$ and $g(x)$ are *equivalent* at a point $a\in\overline{\mathbb{R}}=[-\infty,+\infty]$ and we write:

$$f(x)\underset{a}{\sim} g(x)$$

if:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=1.$$

***Remark 13***. If $f(x)\underset{\infty}{\sim} g(x)$ we simply write: $f(x)\sim g(x)$.

***Remark 14***. The order of the elementary functions at infinity ($x\to\infty$):

$$\ln x \ll x^k \ll a^x \ll x^x$$

where: $k>0, a>1$,
which imply fundamental limits such:

$$\lim_{n\to\infty}\frac{\ln x}{x^k}=0,\lim_{n\to\infty}\frac{x^k}{a^x}=0,\ \lim_{n\to\infty}\frac{a^x}{\ln x}=\infty \text{ etc.}$$

***Theorem 2***. For two series with positive terms:

$$x_n\sim y_n \Rightarrow \sum x_n\sim\sum y_n\ .$$

*Proof*. This is a particular case written in the language of *equivalents* for the main case a) of Theorem 1.
If $x_n\sim y_n$, then $\lim\limits_{n\to\infty}\frac{x_n}{y_n}=1$.
Thus, the two series have the same nature:

$$\Rightarrow \sum x_n\sim\sum y_n$$

***Theorem 3.*** Limit comparison criterion for improper integrals defined on an infinite interval (Nicolescu et al., 1971).
Let $f$ and $g$ be two positive and integrable functions with $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)}=l$.
Then if:

$a)\ l\in(0,\infty)\Rightarrow \int_a^\infty f(x)dx\sim\int_a^\infty g(x)dx$

The two integrals have the same nature.

$b)\ l=0$

$$\Rightarrow\begin{cases} a)\ \int_a^\infty g(x)dx\ conv \Rightarrow \int_a^\infty f(x)dx\ conv \\ b)\ \int_a^\infty f(x)dx\ div \Rightarrow \int_a^\infty g(x)dx\ div \end{cases}$$

$c)\ l=\infty$

$$\Rightarrow\begin{cases} a)\ \int_a^\infty f(x)dx\ conv \Rightarrow \int_a^\infty g(x)dx\ conv \\ b)\ \int_a^\infty g(x)dx\ div \Rightarrow \int_a^\infty f(x)dx\ div \end{cases}$$

***Remark 15***. A similar criterion can be defined at $-\infty$. We leave that to the reader.

***Remark 16***. The theorem is valid even if $f$ and $g$ are positive on a neighbourghood of $\infty$.

***Theorem 4***. The limit comparison criterion with equivalents for improper integrals is defined on an infinite interval.
Let $f$ and $g$ be two positive and integrable functions. If $f(x)\underset{\infty}{\sim} g(x)$, then:

$$\int_a^\infty f(x)dx\sim\int_a^\infty g(x)dx$$

The integrals have the same nature.

*Proof*. The fact that $f(x)\sim_\infty g(x)$ means that:

$$l=\lim_{x\to\infty}\frac{f(x)}{g(x)}=1$$

And we apply case a) of the Theorem 3.

***Theorem 5.*** Limit comparison criterion for improper integrals defined on a finite interval.
Let $f$ and $g$ be two positive and integrable functions with $\lim\limits_{\substack{x\to a\\ x>a}}\frac{f(x)}{g(x)}=l$.
Then if:

$a)\ l\in(0,\infty)\Rightarrow \int_{a+}^{b} f(x)dx\sim\int_{a+}^{b} g(x)dx$

The two integrals have the same nature.

$b)\ l=0$

$$\Rightarrow\begin{cases} a)\ \int_{a+}^{b} g(x)dx\ conv \Rightarrow \int_{a+}^{b} f(x)dx\ conv \\ b)\ \int_{a+}^{b} f(x)dx\ div \Rightarrow \int_{a+}^{b} g(x)dx\ div \end{cases}$$

$c)\ l=\infty$

$$\Rightarrow\begin{cases} a)\ \int_{a+}^{b} f(x)dx\ conv \Rightarrow \int_{a+}^{b} g(x)dx\ conv \\ b)\ \int_{a+}^{b} g(x)dx\ div \Rightarrow \int_{a+}^{b} f(x)dx\ div \end{cases}$$

***Remark 17.*** A similar criterion can be defined at $b-$.

***Theorem 6***. Limit comparison criterion with equivalents for improper integrals defined on a finite interval.
Let $f$ and $g$ be two positive and integrable functions:

i) $f(x) \underset{a}{\sim} g(x) \Rightarrow \int_{a+}^{b} f(x)dx \sim \int_{a+}^{b} g(x)dx$

ii) $f(x) \underset{b}{\sim} g(x) \Rightarrow \int_{a}^{b-} f(x)dx \sim \int_{a}^{b-} g(x)dx$

The integrals have the same nature.

## RESULTS AND DISCUSSIONS

**Applications for series of the limit comparison criteria with equivalents**

*Example 1*. Find the nature of the series:

$$\sum_{n\geq 1} \frac{1}{2\sqrt{n}+1}$$

*Solution*. By Theorem 2

$$\sum_{n\geq 1} \frac{1}{2\sqrt{n}+1} \sim \sum_{n\geq 1} \frac{1}{2\sqrt{n}} = \frac{1}{2} \underbrace{\sum_{n\geq 1} \frac{1}{n^{\frac{1}{2}}}}_{\substack{div \\ (\alpha=\frac{1}{2}<1)}}$$

*Example 2*. Study the convergence of the series:

$$\sum_{n\geq 1} \frac{2n+n^2}{n^5-2n^3+1}$$

*Solution*.

$$\sum_{n\geq 1} \frac{2n+n^2}{n^5-2n^3+1} \sim \sum_{n\geq 1} \frac{n^2}{n^5} = \underbrace{\sum_{n\geq 1} \frac{1}{n^3}}_{\substack{conv \\ (\alpha=3>1)}}$$

*Example 3*.

$$\sum_{n\geq 2} \frac{n-2}{\sqrt{n^2+1}}$$

*Solution*.

$$\sum_{n\geq 2} \frac{n-2}{\sqrt{n^2+1}} \sim \sum_{n\geq 2} \frac{n}{\sqrt{n^2}} = \sum_{n\geq 2} \frac{n}{n} = \sum_{n\geq 1} 1$$

$$= 1+1+\cdots = \infty \ (div)$$

*Example 4*.

$$\sum_{n\geq 2} \frac{n+7}{n(n+1)(n+2)}$$

*Solution*.

$$\sum_{n\geq 1} \frac{n+7}{n(n+1)(n+2)} \sim \sum_{n\geq 1} \frac{n}{n\cdot n\cdot n} = \underbrace{\sum_{n\geq 1} \frac{1}{n^2}}_{\substack{conv \\ (\alpha=2>1)}}$$

The equivalence criterion can be mixed with other criteria, as in:

*Example 5*.

$$\sum_{n\geq 1} \frac{n+3\sqrt{n}+1}{2^n+\ln n}$$

*Solution*.

$$\sum_{n\geq 1} \frac{n+3\sqrt{n}+1}{2^n+\ln n} \sim \sum_{n\geq 2} \frac{n}{2^n}$$

Now, we leave to the reader to show that the series of the equivalent is convergent via *D'Alembert's quotient criterion (the ratio test)*:

$If \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = l$

$$\Rightarrow \begin{cases} l<1 \Rightarrow \sum x_n \quad conv \\ l>1 \Rightarrow \sum x_n \quad div \\ l=1 \Rightarrow ? \ (apply\ other\ criteria\ ) \end{cases}$$

In are some situations, the application of the equivalence criteria is surprising.

*Example 6*.

$$\sum_{n\geq 1} \ln\left(\frac{n+1}{n}\right)$$

*Solution*. Because $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$, it results:

$$\ln(1+x) \underset{0}{\sim} x$$

Taking $x = \frac{1}{n} \to 0$, we obtain:

$$\ln\left(1+\frac{1}{n}\right) \sim \frac{1}{n}$$

Therefore,

$$\sum_{n\geq 1} \ln\left(1+\frac{1}{n}\right) \sim \underbrace{\sum_{n\geq 1} \frac{1}{n}}_{\substack{div \\ (\alpha=1\leq 1)}}$$

Sometimes, applying the equivalence criterion is impossible, as in:
*Example 7*.

$$\sum_{n\geq 2}\frac{n^2}{2^{3n+5}}$$

***Remark 18***. $\frac{n^2}{2^{3n+5}}$ is not equivalent with $\frac{n^2}{2^{3n}}$, because their quotient does't tend to 1.
*Solution*. The series is written in minimal form, so no simpler equivalent can be found (it is self-equivalent!). We leave to the reader to prove that the series is convergent using D'Alembert's criterion.

***Remark 19***. Equivalents can successfully be applied situations such:

$$\sum_{n\geq 2}\frac{F(n)}{G(n)},\quad \sum_{n\geq 2}\left(\frac{F(n)}{G(n)}\right)^k,\quad \sum_{n\geq 2}\sqrt[k]{\frac{F(n)}{G(n)}}$$

where:
- $F(n)$și $G(n)$ are polynomials, and $k \in \mathbb{N}$ is fixed, but it cannot be applied, for example, for:

$$\sum_{n\geq 2}\left(\frac{F(n)}{G(n)}\right)^k$$

because $1^{\infty}$ is indetermination case for function limits.

**Applications for improper integrals of the limit comparison criteria with equivalents**

*Example 1*. Find the nature of the integral:

$$\int_1^{\infty}\frac{x}{x^5+2x^4-x^3+7}dx$$

*Solution*. We apply Theorem 4:
$\int_1^{\infty}\frac{x}{x^5+2x^4-x^3+7}dx \sim \int_1^{\infty}\frac{x}{x^5}dx = \int_1^{\infty}\frac{1}{x^4}dx$
(conv, $\alpha = 4 > 1$).

***Remark 20***. Be easy calculus, it can be shown that:

$$\int_1^{\infty}\frac{1}{x^{\alpha}}dx = \begin{cases} conv, & \alpha > 1 \\ \infty\ (div), & \alpha \leq 1 \end{cases}$$

*Example 2*.
$\int_0^{\infty}\frac{1}{\sqrt{x^2+1}}dx \sim \int_1^{\infty}\frac{1}{\sqrt{x^2+1}}dx \sim \int_1^{\infty}\frac{1}{\sqrt{x^2}}dx = \int_1^{\infty}\frac{1}{x}dx$
(div, $\alpha = 1 \leq 1$).

*Example 3*.
$\int_3^{\infty}\frac{(x+1)^2}{x^4+\sin^2 x}dx \sim \int_3^{\infty}\frac{x^2}{x^4}dx \sim \int_3^{\infty}\frac{1}{x^2}dx$
(conv, $\alpha = 2 > 1$).

***Remark 21***. Be easy calculus, it can be shown that:
i) $\int_{a+}^{b}\frac{1}{(x-a)^{\lambda}}dx = \begin{cases} conv, & \lambda < 1 \\ \infty\ (div), & \lambda \geq 1 \end{cases}$
ii) $\int_{a}^{b-}\frac{1}{(b-x)^{\lambda}}dx = \begin{cases} conv, & \lambda < 1 \\ \infty\ (div), & \lambda \geq 1 \end{cases}$
*Example 4*. Study the convergence:

$$\int_0^1\frac{1}{\sqrt{1-x^2}}dx.$$

*Solution*. We apply Theorem 6:

$$\int_0^1\frac{1}{\sqrt{1-x^2}}dx = \int_0^1\frac{1}{\sqrt{(1-x)(1+x)}}dx$$

$$\underset{1}{\overset{\square}{\sim}}\int_0^1\frac{1}{\sqrt{(1-x)(1+1)}}dx = \frac{1}{\sqrt{2}}\int_0^1\frac{1}{(1-x)^{\frac{1}{2}}}dx$$

(conv, $\lambda = \frac{1}{2} < 1$).

***Remark 22***. The integral is improper because the numerator is undefined at 1. Generally, authors don't specify the limit to the right or to the left, it can be deduced from the context.
*Example 5*.

$$\int_1^3\sqrt{\frac{1}{(3-x)^5(x-1)}}dx.$$

*Solution*. $\int_1^3\sqrt{\frac{1}{(3-x)^5(x-1)}}dx =$

$$\underbrace{\int_1^2\sqrt{\frac{1}{(3-x)^5(x-1)}}dx}_{I_1} + \underbrace{\int_2^3\sqrt{\frac{1}{(3-x)^5(x-1)}}dx}_{I_2}$$

We have:

$$\sqrt{\frac{1}{(3-x)^5(x-1)}}\underset{1}{\overset{\square}{\sim}}\sqrt{\frac{1}{(3-1)^5(x-1)}} = \frac{1}{4\sqrt{2}}\cdot\sqrt{\frac{1}{x-1}}$$

$$= \frac{1}{4\sqrt{2}}\cdot\frac{1}{(x-1)^{\frac{1}{2}}}$$

Thus:

$$I_1 = \int_1^2\sqrt{\frac{1}{(3-x)^5(1-x)^3}}dx \sim \int_2^3\frac{1}{(x-1)^{\frac{1}{2}}}dx,$$

(conv , $\lambda = \frac{1}{2} < 1$).

We have:

$$\sqrt{\frac{1}{(3-x)^5(x-1)}}\underset{3}{\overset{\square}{\sim}}\sqrt{\frac{1}{(3-x)^5(3-1)}} = \frac{1}{\sqrt{2}}\cdot\sqrt{\frac{1}{(3-x)^5}}$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{(3-x)^{\frac{5}{2}}}$$

$$I_2 = \int_2^3 \sqrt{\frac{1}{(3-x)^5(1-x)^3}}\,dx \sim \int_2^3 \frac{1}{(3-x)^{\frac{5}{2}}}\,dx,$$

$$(\text{div} = \infty,\ \lambda = \frac{5}{2} > 1).$$

Therefore,

$$I = I_1 + I_2 = conv + \infty = \infty\ (div)$$

**Some applications of series or improper integrals in or outside mathematics**

**Zeno's arrow paradox in philosophy**

*Zeno of Elea* (490-430 b.Ch.) was a pre-Socratic greek philosopher.

In the arrow paradox (Huggett, 2024), Zeno states that for motion to occur, an object must change its position. He gives an example of an arrow in flight (Figure 1). He states that at instant of time, the arrow is neither moving to where it is, nor to where it is not. It cannot move to where it is not, because no time elapses for it to move there; it cannot move to where it is, because it is already there. In other words, at every instant of time there is no motion occurring. If everything is motionless at every instant, and time is entirely composed of instants, then motion is impossible.

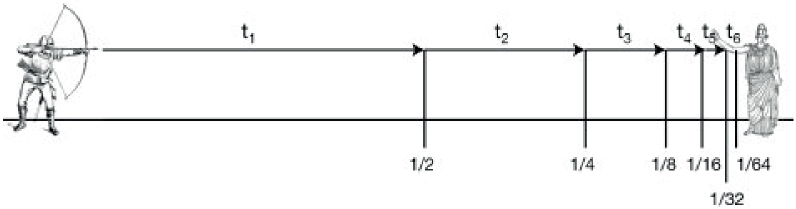


Figure 1. Zeno's paradox
http://www.naturphilosophie.co.uk/zenos-paradoxes-or-what-happened-when-achilles-and-the-hare-decided-to-outfox-the-legendary-tortoise

In fact, this paradox shows that time is infinitely divisible, and can be better understood with the series:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 1$$

which explains that, when adding all the segments, the arrow reaches its target.

**Taylor series**

Every elementary function can be locally approximated around the point *a* by a sum of its derivatives of the form:

$$f(x) = \sum_{n \geq 0} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

For example:

$$sin\, x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots, \qquad x \in \mathbb{R}$$

Taking a sufficient number of terms, one may approximate any functions at a point with a convenient number of decimals.

Taylor series are the main tool for approximating the values of elementary functions on any ordinary scientific calculator, although there are some additional methods of improving the accuracy.

**Graphical interpretation of Riemann and improper integrals**

It is well known that the Riemann integral of a positive function $f: [a, b] \to \mathbb{R}$ is the are of the subgraph above the abscissa. This property can also be extended to improper integrals. Thus, an important role of integrals is in calculating areas, volumes, and lengths.

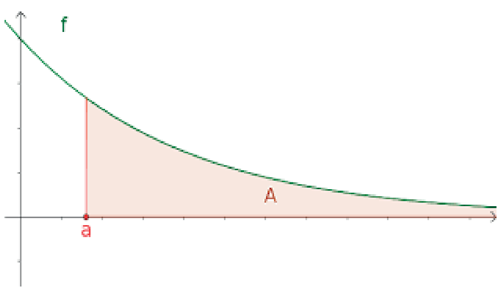


Figure 2. Geometric interpretation of the improper integral

The area of the subgraph of a positive function represented in Figure 2 is:

$$A = \int_{a}^{\infty} f(x)dx$$

For example, in physics, if $f(x)$ represents the value of a force, the integral represents the mechanical work of the force.

**Fourier series**

They were first used in the early 1800's by Joseph Fourier in order to find solutions to the heat equation.

Nowadays they have applications in physics, signal processing, image processing, conversion of special data into frequency data etc. (Budău et al., 2023; Calmuc et al., 2022).

A usual Fourier series is a representation of a periodic function $f(x)$ on $[-\pi, \pi]$ as a series of sines and cosines of the form:

$$f(x) = \frac{1}{2}a_0 + \sum_{n\geq 1} a_n \cos nx + \sum_{n\geq 1} b_n \cos nx$$

The coefficients have the following formulas:

$$a_0 = \int_{-\infty}^{\infty} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

## CONCLUSIONS

Integrals and series play an important role in, or outside mathematics and they are closely related to the overall progress of science and humanity. The limit comparison criteria with equivalents for series with positive terms or improper integrals are very useful and natural techniques. Of course, as other convergence criteria, they can't be used for computing sums or integrals in case of convergence.

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