Scientific Papers. Series E. Land Reclamation, Earth Observation & Surveying, Environmental Engineering. Vol. XIV, 2025
Print ISSN 2285-6064, CD-ROM ISSN 2285-6072, Online ISSN 2393-5138, ISSN-L 2285-6064

FIBONACCI'S SEQUENCE IN NATURE, SCIENCE AND ARTS. THE GOLDEN RATIO

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Abstract

Leonardo Fibonacci (c. 1170 - c. 1240-1250), an Italian mathematician often regarded as the most gifted European mathematician of the Middle Ages, played a crucial role in introducing the Hindu–Arabic numeral system to Europe. His seminal work, Liber Abaci (1202), not only promoted this numerical framework but also featured the now-famous Fibonacci sequence - an integer series with deep mathematical properties and widespread applications. This paper explores the mathematical foundations of the Fibonacci sequence, including its recurrence relations, algebraic and matrix representations, and connections to continuous fractions. Furthermore, it examines the relationship between the sequence and the golden ratio (φ), an irrational number often referred to as the "divine proportion". The golden ratio is linked to aesthetically pleasing proportions and appears in various domains such as art, architecture, music, biology, and cosmology. Through historical analysis and illustrative examples, this work highlights the enduring influence of Fibonacci's legacy and the remarkable intersection between mathematical theory and patterns observed in the natural and cultural world.

***Key words**: Fibonacci sequence, Fibonacci numbers, golden ratio, irrational numbers, recursive functions, mathematical modelling, patterns in nature, mathematical aesthetics, interdisciplinary applications.*

"Everything that is correct thinking is either mathematics or susceptible to mathematization"
Grigore Moisil

INTRODUCTION

Mathematics, logic, philosophy, astronomy, and cosmology form the foundation of many of the principles, calculations, and models used in contemporary science and daily life. These disciplines have shaped humanity's understanding of the world and continue to provide tools for discovery and innovation.

Among the great contributors to mathematical thought, names such as Pythagoras, Archimedes, Euler, and Fibonacci stand out. Their works laid the groundwork for theories and methods that remain relevant today. One particularly influential figure is Leonardo Fibonacci (c. 1170 - c. 1240-1250), also known as Leonardo of Pisa, a mathematician widely regarded as the most talented of the Western Middle Ages.

Fibonacci was the son of Guglielmo, an Italian merchant and customs official stationed in Bugia (modern-day Béjaïa, Algeria). Accompanying his father on commercial journeys throughout the Mediterranean, Fibonacci was exposed to diverse mathematical practices. It was in Bugia that he encountered the Hindu-Arabic numeral system - a revolutionary system far superior to the Roman numerals then used in Europe.

The name "Fibonacci" is derived from the Latin *filius Bonacci*, meaning "son of Bonacci". The word "Bonacci" likely comes from the Latin *bonus*, meaning "good," suggesting "fortunate son". This name would eventually become synonymous with one of the most elegant numerical sequences in mathematics.

This paper provides an overview of Fibonacci's mathematical legacy, focusing on his introduction of the Hindu–Arabic numeral system to Europe and the presentation of the Fibonacci sequence in his seminal work *Liber Abaci*. It also explores the mathematical properties of this sequence and its connection to the golden ratio - a number that has fascinated mathematicians, artists, and scientists for centuries due to its unique properties and surprising appearances in both natural and

Print ISSN 2285-6064, CD-ROM ISSN 2285-6072, Online ISSN 2393-5138, ISSN-L 2285-6064

human-made structures (Livio, 2003; Wang & Johnson, 2008; Choi et al., 2023).

FIBONACCI'S MATHEMATICAL HERITAGE

A pivotal milestone in Fibonacci's mathematical legacy was his role in introducing and promoting the Hindu–Arabic numeral system in Europe. This achievement was primarily realized through the publication of his influential work, *Liber Abaci* (The Book of Calculation), in the early 13th century (Beebe, 2009).

It is important to emphasize that Fibonacci's contributions were made in an era without modern computational tools such as calculators or computers. The introduction of an efficient numerical system was therefore revolutionary, facilitating arithmetic operations that were previously cumbersome under the Roman numeral system.

Among the many examples in *Liber Abaci*, Fibonacci included a now-famous problem involving the growth of a rabbit population. This example led to the popularization of the integer sequence that would later bear his name—the Fibonacci sequence. Although Fibonacci did not invent the sequence, his use of it in this context significantly contributed to its recognition and enduring legacy in mathematical literature.

Liber Abacci

Fibonacci employed a unique fractional notation in *Liber Abaci*, using composite fractions in which a sequence of numerators and denominators was written beneath a single fraction bar. A notable example is found in a manuscript preserved at the National Central Library, where Fibonacci lists the numbers 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, and 377—forming the well-known Fibonacci sequence (Boyer, 1968).

This page, shown in Figure 1, also illustrates how certain digits - particularly 2, 8, and 9 - bear a stronger resemblance to modern Arabic numerals than to their Eastern Arabic or Indian counterparts. While these early numbers are simple to compute or recall, the Fibonacci sequence extends far beyond its initial terms, revealing complex and profound mathematical properties (Sigler, 2002).

Liber Abaci - Latin for "The Book of Calculation" - was published in 1202 and remains one of the most influential arithmetic texts of the medieval period (Devlin, 2012). Although the original edition no longer survives, a revised version was completed in 1227 and dedicated to the scholar Michael Scot. At least nineteen manuscripts containing portions of this work still exist, including three complete versions from the 13th and 14th centuries and nine known incomplete copies dating from the 13th to 15th centuries (Mollin, 2002).

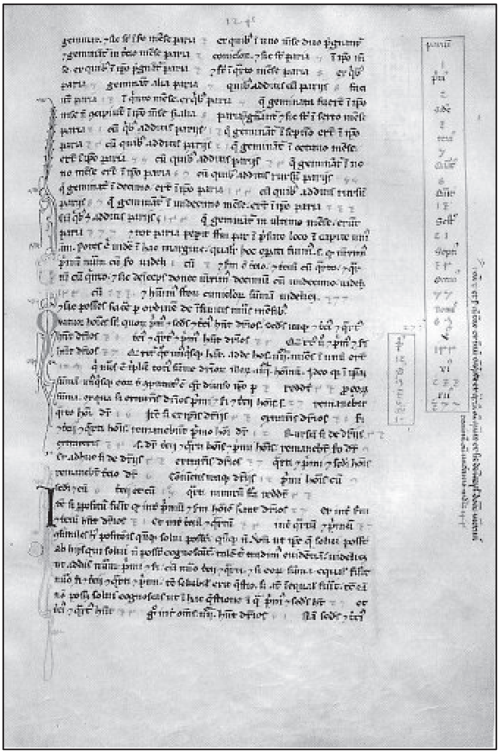

Figure 1. A page from *Liber Abaci*, displaying early use of the Fibonacci sequence and Arabic numeral forms (Pisano & Bussotti, 2015)

The first printed edition of *Liber Abaci* appeared in 1857, translated into Italian by Boncompagni. A complete English translation did not appear until the early 21st century, highlighting the enduring interest in Fibonacci's contributions.

This work is particularly renowned for introducing the base-10 positional system and Arabic numeral symbols to Europe, replacing the cumbersome Roman system. Fibonacci emphasized the practical superiority of this approach throughout the text.

Beyond arithmetic, *Liber Abaci* demonstrated the widespread presence of Fibonacci numbers in nature. These numbers appear in spiraling arrangements of plant growth, such as sunflower seed patterns, pinecones, and other botanical structures. Similar spirals are observed in non-living systems such as hurricanes, water vortices, and galaxies. Additionally, biological structures like snail shells, nautilus spirals, the cochlea of the inner ear, and even goat horns often follow logarithmic spiral patterns aligned with the Fibonacci sequence.

Significantly, *Liber Abaci* was the first Western book to introduce and promote the Hindu–Arabic numeral system, including symbols resembling modern digits. The work made a compelling case for adopting this new system, explaining its advantages for both theoretical and practical calculations.

Although the title is sometimes mistranslated as *The Book of the Abacus*, Sigler (2002) clarifies that the term *abacus* during Fibonacci's time referred to "calculation" in general, not specifically to the counting device. In fact, in medieval Italy, the spelling *abbacus* (with two "b"s) denoted computations using Hindu–Arabic numerals, thus avoiding confusion.

Liber Abaci outlined techniques for performing arithmetic without relying on the traditional abacus. Ore (1948) noted a long-standing rivalry that persisted after the book's publication—between *algorists*, who promoted the new numerical system, and *abacists*, who remained loyal to the Roman numeral-based counting board.

Mathematics historian Carl Boyer emphasized that *Liber Abaci*, while not a treatise on the abacus itself, is "a very thorough treatise on algebraic methods and problems in which the use of the Hindu–Arabic numerals is strongly advocated" (Boyer, 1968).

The book is organized into four main sections:

- Section I introduces the Hindu–Arabic numeral system, arithmetic operations, and methods for converting between numerical representations. It also includes the earliest known use of trial division for identifying and factoring composite numbers.
- Section II applies arithmetic to commercial problems, including currency conversion, measurements, and interest calculations.
- Section III explores more advanced mathematical topics, such as the Chinese Remainder Theorem, perfect numbers, Mersenne primes, arithmetic series, and square pyramidal numbers. This section also presents the famous rabbit problem, which popularized the Fibonacci sequence.
- Section IV discusses numerical and geometric approximations of irrational numbers, such as square roots.

Notably, *Liber Abaci* contains several Euclidean geometric proofs, revealing Fibonacci's familiarity with classical Greek mathematics. Furthermore, his algebraic problem-solving methods suggest influence from earlier scholars such as the 10th-century Egyptian mathematician Abū Kāmil Shujāʿ ibn Aslam (Mollin, 2002).

Fibonacci's notation for fractions (Moyon et al., 2015)

In reading *Liber Abaci*, it is essential to understand Fibonacci's unique notation for rational numbers. His system represents a transitional form between ancient Egyptian fractions - commonly used up to that time - and the modern fractional notation still in use today. Fibonacci's approach differs from contemporary notation in several keyways:

1. *Order of mixed numbers:* modern notation typically places the fractional part to the right of the whole number (e.g., $2\frac{1}{5}$ for $\frac{11}{5}$). In contrast, Fibonacci wrote the fraction first, followed by the whole number: for instance, he would write "$\frac{1}{5}2$" to represent $2\frac{1}{5}$

2. *Composite fractions:* Fibonacci used a form of composite fraction notation in which a single fraction bar encompassed multiple numerators and denominators. Each term represented a separate fraction, where the numerator was divided by the product of all denominators to its right. For example:

$$\frac{b\,a}{d\,c} = \frac{a}{c} + \frac{b}{cd}, \text{ and } \frac{c\,b\,a}{f\,e\,d} = \frac{a}{d} + \frac{b}{de} + \frac{c}{def}.$$

As a specific example, the fraction, $\frac{27}{30}$ could be represented as $\frac{1\,1\,4}{2\,3\,5}$, which translates to:

$$\frac{4}{5} + \frac{1}{3\cdot 5} + \frac{1}{2\cdot 3\cdot 5}.$$

This form of mixed radix notation was particularly useful for handling weights, measures, and currencies. Unlike traditional mixed-radix systems that often-omitted denominators, Fibonacci's notation explicitly included them, allowing more precision and adaptability for various practical contexts.

3. *Additive notation:* Fibonacci occasionally wrote multiple simple fractions side by side to indicate their sum. For example, since $\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$, so a notation like $\frac{1}{2}\,\frac{1}{3}$ 4 would represent the mixed number $4\frac{5}{6}$, or simply the ordinary fraction $\frac{29}{6}$.

This form of notation can be distinguished from composite fractions by a visual break in the fraction bar. Moreover, when all numerators are 1 and the denominators are distinct, the result corresponds to an *Egyptian fraction* - a sum of distinct unit fractions.

Fibonacci sometimes combined additive and composite notations to represent numbers more flexibly. His system allowed for multiple representations of the same value, and *Liber Abaci* includes several methods for converting between these forms.

Notably, Chapter II.7 of *Liber Abaci* features a collection of techniques for expressing improper fractions as sums of unit fractions. Among these is the *greedy algorithm* - a method later referred to as the *Fibonacci–Sylvester expansion* - which selects the largest possible unit fraction at each step to construct the sum (O'Connor et al., n.d.).

Modus Indorum – The Method of the Indians

In *Liber Abaci*, Fibonacci introduces the concept of the *Modus Indorum* - Latin for "method of the Indians" - which refers to what is now known as the Hindu–Arabic numeral system or base-10 positional notation. This system, developed in India and transmitted to the Islamic world, was recognized by Fibonacci for its superior efficiency in calculation and record-keeping.

The text includes a presentation of the nine Indian numerals: 1, 2, 3, 4, 5, 6, 7, 8, 9 and introduces the symbol 0, referred to by the Arabs as *zephyrum* (from the Arabic *ṣifr*), meaning "empty" or "zero". Using these ten symbols, any number can be written through positional value - a concept that was revolutionary in contrast to the Roman numeral system then prevalent in Europe.

Thus, *Liber Abaci* offers not only a practical guide to arithmetic but also a comprehensive introduction to the use of digits 0 through 9, along with the principles of place value, which lie at the core of modern number systems.

Prior to the adoption of this system, Europe relied on Roman numerals, which lacked a symbol of zero and made complex arithmetic operations - such as multiplication, division, and algebraic problem-solving - extremely difficult, if not impossible. As a result, the methods of modern mathematics could not fully develop.

Fibonacci's endorsement of the Hindu–Arabic system marked a turning point in European mathematics. Although *Liber Abaci* was instrumental in initiating this transition, the widespread adoption of the system was gradual and extended over several centuries. As Ore (1948) notes, the process was "long-drawn-out", and it was not until the end of the 16th century that the Hindu–Arabic numeral system became widely accepted across Europe.

Some Mathematical Properties of Fibonacci Numbers

The Fibonacci sequence is a sequence of natural numbers defined by the recurrence relation (Vorobiev et al., 2002):

$$F_1 = 1,\ F_2 = 1$$

$$F_n = F_{n-1} + F_{n-2}, \qquad n \geq 3$$

The recurrence relation has a characteristic equation:

$$r^2 = r + 1$$

Solving this quadratic yields the roots:

$$r_1 = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad r_2 = \frac{1-\sqrt{5}}{2}$$

Remark, the first root,

$$\varphi = r_1 = \frac{1+\sqrt{5}}{2}$$

is known as the **Golden Ratio**.

Using these roots, the general term of the Fibonacci sequence can be expressed as:

$$F_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

where $c_1, c_2 \in \mathbb{R}$ are the constants.

Applying the initial conditions

$$F_1 = 1, \qquad F_2 = 1,$$

we find:

$c_1 = \frac{1}{\sqrt{5}}$ and $c_2 = -\frac{1}{\sqrt{5}}$,

so, the *general therm* of Fibonacci's sequence is:

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Matrix form

The Fibonacci sequence can also be expressed in matrix form:

$$\begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$$

If we denote:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

we obtain

$$\begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix} = A^n \begin{pmatrix} F_2 \\ F_1 \end{pmatrix}, n \geq 1.$$

then:

$$A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

Continuous fractions

The golden ratio can be represented as an infinite continued fraction:

$$\varphi = 1 + \frac{1}{\varphi} = \cdots$$

$$= 1 + \frac{1}{1 + \frac{1}{1+\frac{1}{1+\cdots}}}$$

Selected identities and properties

1. Relation between φ and Fibonacci numbers:

$$\varphi^n = F_n\varphi + F_{n-1}, \qquad n \geq 1$$

Proof: by induction.

For *n = 2* the equality becomes:

$$\varphi^2 = F_2\varphi + F_1 \Leftrightarrow \varphi^2 = \varphi + 1 \text{ (true)}.$$

$n \to n+1$

$$\varphi^n = F_n\varphi + F_{n-1} | \cdot \varphi \Longrightarrow$$

$$\varphi^{n+1} = F_n\varphi^2 + F_{n-1}\varphi =$$

$$F_n(\varphi+1) + F_{n-1}\varphi = (F_n + F_{n-1})\varphi + F_n = F_{n+1}\varphi + F_n$$

Inductive step follows by multiplying both sides by φ and simplifying.

2. Cassini's identity:

$$F_{n+1} \cdot F_{n-1} - F_n^2 = (-1)^n$$

Proof. We consider *m = 1* in the equality from below. A more elegant proof is via matrices.

We consider the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ which characterizes Fibonacci's sequence and knowing $\det A = -1$ we obtain this identity.

$$A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

3. Catalan's Identity (Generalization of Cassini's):

$$F_{n+m} \cdot F_{n-m} - F_n^2 = (-1)^{n-m+1}F_m^2$$

Proof. By induction, we leave this to the reader.

4. Addition identity:

$$F_mF_{n+1} + F_{m-1}F_n = F_{m+n}$$

Proof. By induction on m,

Example cases:

- m = 2

$$F_2F_{n+1} + F_1F_n = F_{n+2} \Leftrightarrow F_{n+1} + F_n = F_{n+2}$$
(true)

- m = 3

$$F_3F_{n+1} + F_2F_n = 2F_{n+1} + F_n = F_{n+1} + F_{n+2} = F_{n+3} \text{ (true)}$$

$m-1, m \to m+1$

Adding the relations:

$F_{m-1}F_{n+1} + F_{m-2}F_n = F_{m+n-1}$ and $F_mF_{n+1} + F_{m-1}F_n = F_{m+n}$

we obtain:

$$(F_{m-1} + F_m)F_{n+1} + (F_{m-2} + F_{m-1})F_n = F_{m+n-1} + F_{m+n}$$

which gives:

$$F_{m+1}F_{n+1} + F_mF_n = F_{m+1+n}$$

5. Sum identity:

$$\sum_{k=1}^{n} F_k = F_{n+2} - 1$$

Proof. By induction.

For $n = 1$ the equality becomes $F_1 = F_2 = 1$ (true).
$n \to n+1$

$$\sum_{k=1}^{n+1} F_k = \sum_{k=1}^{n} F_k + F_{n+1} = F_{n+2} + F_{n+1} - 1 = F_{n+3} - 1$$

6. Pythagorean identity:
Every second term in the Fibonacci sequence starting from F_5 forms the hypotenuse of a right triangle:

$$(F_n F_{n+3})^2 + (2F_{n+1}F_{n+2})^2 = F_{2n+3}^2$$

Proof left to the reader.
7. Odd-Index Sum Identity:

$$\sum_{k=0}^{n-1} F_{2k+1} = F_{2n}$$

Proof. By induction. For n=1 the equality becomes $F_1 = F_2 = 1$ (true).
$n \to n+1$

$$\sum_{k=0}^{n} F_{2k+1} = \sum_{k=0}^{n-1} F_{2k+1} + F_{2n+1} = F_{2n} + F_{2n+1} = F_{2n+2}$$

8.

$$\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$$

Proof. By induction. For $n = 1$ the equality becomes $F_1^2 = F_1 F_2 = 1$ (true).
$n \to n+1$

$$\sum_{k=0}^{n+1} F_k^2 = \sum_{k=0}^{n} F_k^2 + F_{n+1}^2 = F_n F_{n+1} + F_{n+1}^2 = F_{n+1}(F_n + F_{n+1}) = F_{n+1}F_{n+2}$$

EXAMPLES OF THE GOLDEN RATIO IN ARTS AND NATURE

Golden ratio

The *golden ratio* is a special irrational number, denoted by φ, and is defined by the division of a line into two parts (Figure 2), a and b, such that:

$$\frac{a}{b} = \frac{a+b}{a}$$

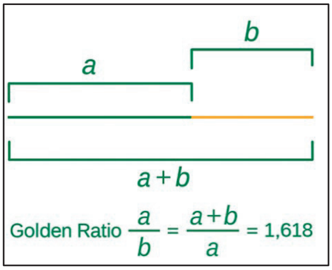

Figure 2. Golden Ratio

Golden Rectangle

A *Golden Rectangle* is a rectangle whose side lengths are in the golden ratio (Figure 3). That is, the ratio of the longer side to the shorter side equals φ.
This geometric figure is known for its aesthetic harmony and has been widely used in art, design, and architecture since antiquity.

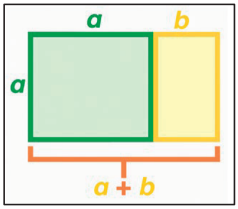

Figure 3. Golden Rectangle

Golden angle

In geometry, the *golden angle* arises when a circle is divided according to the golden ratio (Figure 4). Specifically, it is the smaller of the two angles formed when the circumference is split such that the ratio of the arc lengths is the golden ratio.
Its exact value is:

$$360^o\left(1 - \frac{1}{\varphi}\right) = \left(3 - \sqrt{5}\right)180^o \approx 137{,}5^o = 2\pi\left(3 - \sqrt{5}\right) rad$$

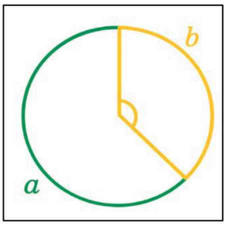

Figure 4. Golden Angle

Scientific Papers. Series E. Land Reclamation, Earth Observation & Surveying, Environmental Engineering. Vol. XIV, 2025
Print ISSN 2285-6064, CD-ROM ISSN 2285-6072, Online ISSN 2393-5138, ISSN-L 2285-6064

Golden spiral

A *golden spiral* is a type of logarithmic spiral whose growth factor is equal to the golden ratio φ. Golden spirals are self-similar, meaning their shape remains unchanged under magnification (Figure 5).

The polar equation of a golden spiral is:

$$r = \varphi^{\frac{2\theta}{\pi}}$$

where:

- r is the radius;
- θ is the angle in radians;
- φ is the golden ratio.

More generally, a logarithmic spiral with growth factor B can be written as:

$$r = A\varphi^{B\theta}$$

which can be rewritten:

$$\theta = \frac{1}{B}\log\left(\frac{r}{A}\right)$$

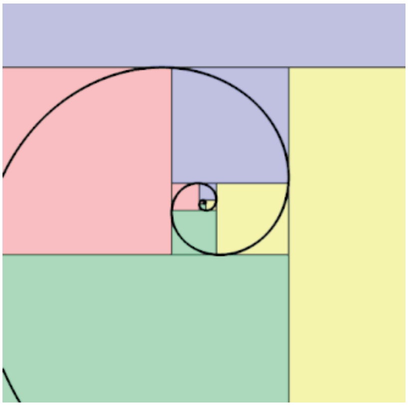

Figure 5. Golden Spiral
(https://en.wikipedia.org/wiki/Golden_spiral)

GOLDEN RATIO IN ARTS

Painting

The golden ratio has played a notable role in the composition of some of the most iconic works of art throughout history. Its presence is believed to enhance visual harmony, balance, and aesthetic appeal, contributing to the emotional and perceptual impact of a painting. Several masterpieces are often cited for their use of this proportion, including: "The Last Supper" and "The Mona Lisa" by Leonardo da Vinci (Figure 6), "The creation of Adam" by Michelangelo, "The Birth of Venus" by Botticelli, "The Starry Night" by Van Gogh, "The Persistence of Memory" by Salvador Dali. In these works, the golden ratio is thought to have guided the spatial arrangement of subjects and background elements, producing a visual equilibrium that resonates with the human sense of proportion and beauty. Artists throughout centuries have adopted the golden ratio, whether consciously or intuitively, as a compositional tool to create visual narratives that are pleasing and memorable. Alongside perspective, symmetry, and contrast, it remains one of the key elements used to achieve artistic coherence.

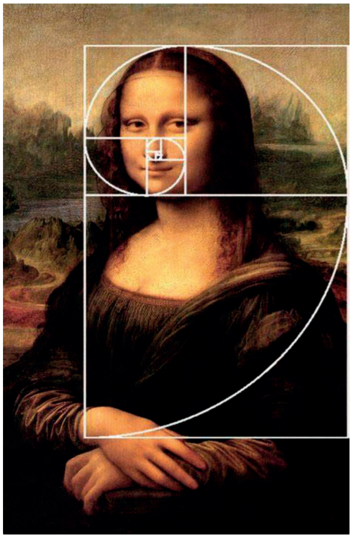

Figure 6. Mona Lisa - a painting frequently associated with golden ratio-based composition
(https://www.bing.com/images/)

Architecture

The golden ratio is also prevalent in architecture, appearing in both ancient monuments and contemporary structures. Long before Fibonacci's time, this proportion was intuitively employed in designs that were considered naturally harmonious and aesthetically pleasing. Examples of structures often associated with the golden ratio include: The "Pantheon" in Rome (Figure 7), The "Great Pyramids" of Giza (Figure 8).

Architects and builders may have used the golden ratio not just for its beauty but also for structural balance and proportion. In modern times, it continues to inform architectural design in buildings, bridges, and even furniture, reflecting humanity's enduring appreciation for mathematical harmony in physical space.

Figure 7. Pantheon - a classical Roman example often associated with golden proportions (https://www.bing.com/images/)

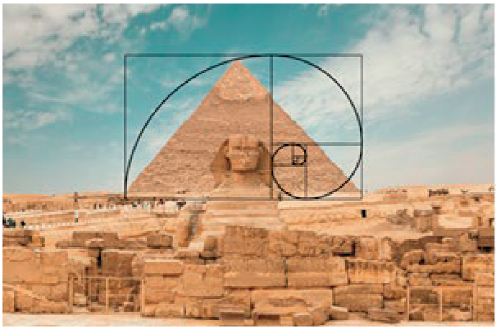

Figure 8. Pyramids - widely analyzed for golden ratio correlations in their dimensions (https://www.bing.com/images/)

Music

The golden ratio has also been observed in the structure and timing of musical compositions. Renowned composers such as Wolfgang Amadeus Mozart, Ludwig van Beethoven, and Claude Debussy are believed to have employed this proportion - either deliberately or intuitively - to shape the form and flow of their works.

For example, in Beethoven's Fifth Symphony, researchers have found structural divisions that align closely with the golden ratio, particularly in the placement of key climactic moments and thematic transitions (Figure 9).

Additionally, the golden ratio has been linked to the design of Stradivarius violins (Figure 10). Studies suggest that certain dimensional proportions in these famous instruments reflect golden ratio principles, potentially contributing to their visual elegance and acoustic excellence.

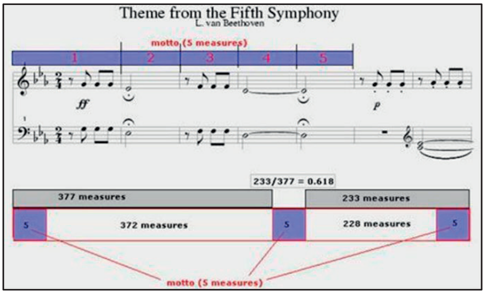

Figure 9. Beethoven's Fifth Symphony - noted for structural divisions consistent with the golden (https://www.bing.com/images/)

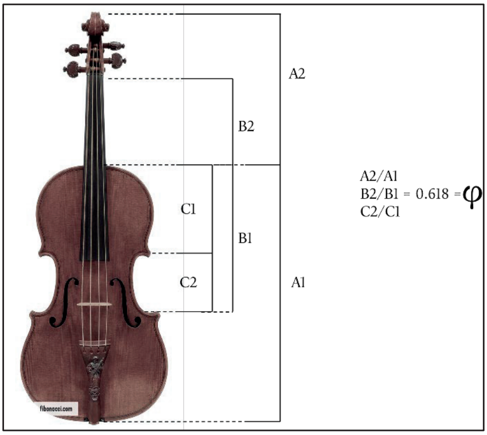

Figure 10. Stradivarius violine - exhibits golden proportions in design (https://www.bing.com/images/)

GOLDEN RATIO IN NATURE

Sunflower seeds

In sunflowers, the seeds located at the center are arranged in spiral patterns that follow Fibonacci numbers (Figure 11). These patterns allow for optimal packing and efficient use of space, maximizing the number of seeds that can fit in each area.

Pinecones

Pinecones frequently exhibit spirals in Fibonacci-related pairs, such as 3 and 5, 5 and 8, or 8 and 13. This form of natural arrangement is part of a broader phenomenon called phyllotaxis - the spiral pattern by which leaves, or other botanical elements are organized around a stem (Figure 12). Similar spiraling patterns are found in the outer petals of artichokes, succulents, and various flower buds.

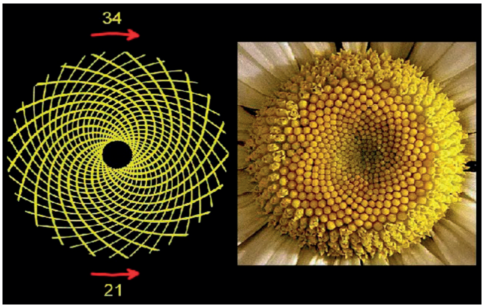

Figure 11. Sunflowers seeds - arranged in spirals that correspond to Fibonacci numbers (https://www.mathnasium.com/blog/golden-ratio-in-nature\)

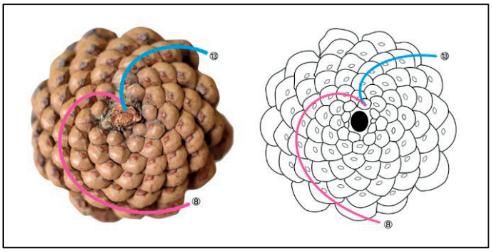

Figure 12. Pinecone with 8/13 configuration (https://craftofcoding.wordpress.com/2022/05/11/fibonacci-and-pinecones/)

Plant leaves

The golden ratio also manifests in the arrangement and structure of plant leaves. To maximize exposure to sunlight, many plants grow their leaves in spiral patterns, minimizing the shadow cast on lower leaves. This spiral spacing often aligns with the golden angle, which helps to optimize photosynthetic efficiency (Figure 13).

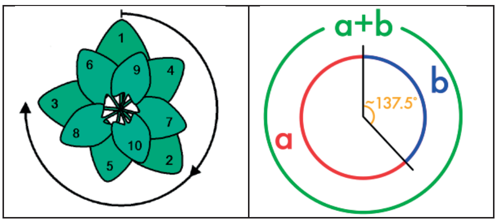

Figure 13. Leaves arrangement - successive leaves are separated by the golden angle (https://www.projectrhea.org/rhea/index.php/MA279Fall2018Topic1_Leaves)

Additionally, internal leaf structures show golden proportions. For instance, the vein spacing in some species approximates the golden ratio, and leaves of the Ginkgo tree often grow with dimensions reflecting this proportion (Figure 14).

Figure 14. Venation of a leaf - illustrating golden ratio spacing (https://www.projectrhea.org/rhea/index.php/MA279Fall2018Topic1_Leaves)

Nautilus Shells

Nautilus shells are often cited as natural illustrations of the golden spiral, a logarithmic spiral whose growth factor is the golden ratio. As the shell grows, it maintains a consistent spiral shape, exemplifying self-similarity - a key characteristic of golden spirals (Figure 15).

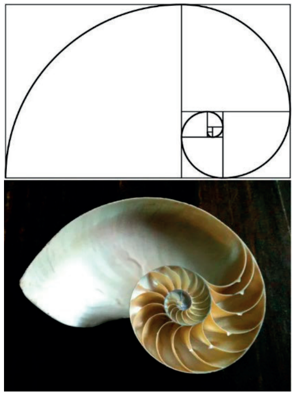

Figure 15. Nautilus shells - a classic example of the golden spiral in nature (https://www.mathnasium.com/blog/golden-ratio-in-nature\)

DNA

The golden ratio appears even at the molecular level. In the structure of DNA, relationships between its geometric features - including the length of a full turn of the helix and the width of the molecule - are often approximated by the golden ratio (Figure 16), suggesting an underlying harmony in the genetic blueprint of life.

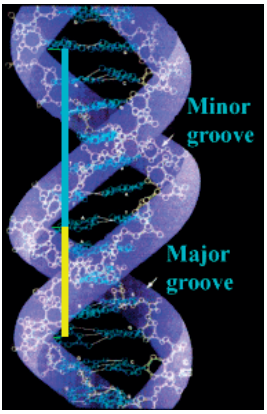

Figure 16. DNA - illustrating proportions that reflect the golden ratio
(https://www.goldennumber.net/dna/)

Fibonacci numbers in tornados, vortices or galaxies?

While many spiral shapes in nature arise from purely physical, non-biological processes - such as whirlpools, vortices in bodies of water, or the swirling formations of hurricane clouds and clear lanes - these spirals do not consistently follow Fibonacci-like patterns in their mathematical structure over time (Figure 17).

It may be possible to capture a snapshot where certain features temporarily exhibit ratios resembling those found in the Fibonacci sequence, but these patterns are neither sustained nor inherent to the structures themselves. In fact, the Fibonacci-like spirals observed in galaxies are more a result of human perception than a fundamental truth of the universe!

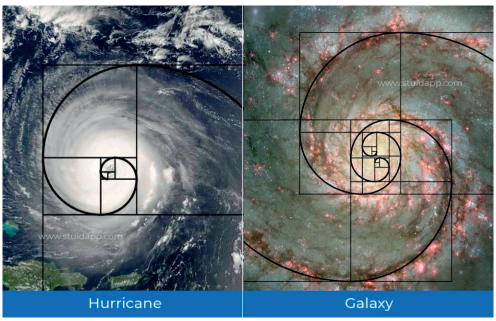

Figure 17. Hurricane and spiral galaxy – natural forms that appear Fibonacci-like, though not mathematically exact
(https://www.bing.com/images/)

CONCLUSIONS

The Fibonacci sequence and the associated golden ratio have long captivated mathematicians, scientists, and artists alike due to their mathematical elegance and their intriguing appearance in a wide range of natural and human-made phenomena.

While their mathematical foundations are firmly rooted in number theory and recurrence relations, their significance extends well beyond abstract theory. In fields such as biology, art, architecture, and music, these concepts often appear - sometimes as a matter of natural optimization, sometimes as a tool for achieving aesthetic harmony.

In nature, the Fibonacci sequence is often observed in plant growth patterns, leaf arrangements, pinecones, and flower petals, where the underlying spiral forms offer practical advantages such as sunlight exposure and space efficiency. The golden spiral appears in natural objects like nautilus shells and hurricanes, though its presence in large-scale phenomena like galaxies is often more interpretive than mathematically precise.

In art and architecture, the golden ratio has been consciously applied to design works that are balanced and pleasing to the eye, from Renaissance masterpieces to ancient monuments and modern constructions. Similarly, in music, some compositions reveal structural proportions aligning with the golden ratio, contributing to their rhythmic and thematic cohesion.

However, it is important to distinguish between intentional use and retrospective attribution. The golden ratio and Fibonacci numbers do not universally govern natural or artistic forms, and their presence is not always exact. In many cases, their appearance is approximate or coincidental and should be appreciated as part of a broader interplay between mathematics and the natural world - not as a universal design code.

In conclusion, the enduring fascination with Fibonacci numbers and the golden ratio lies in their ability to bridge pure mathematics with observable reality. Whether in a sunflower, a symphony, or a spiral staircase, they offer a compelling reminder of mathematical patterns that often underline beauty, function, and structure in both nature and human creativity.

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